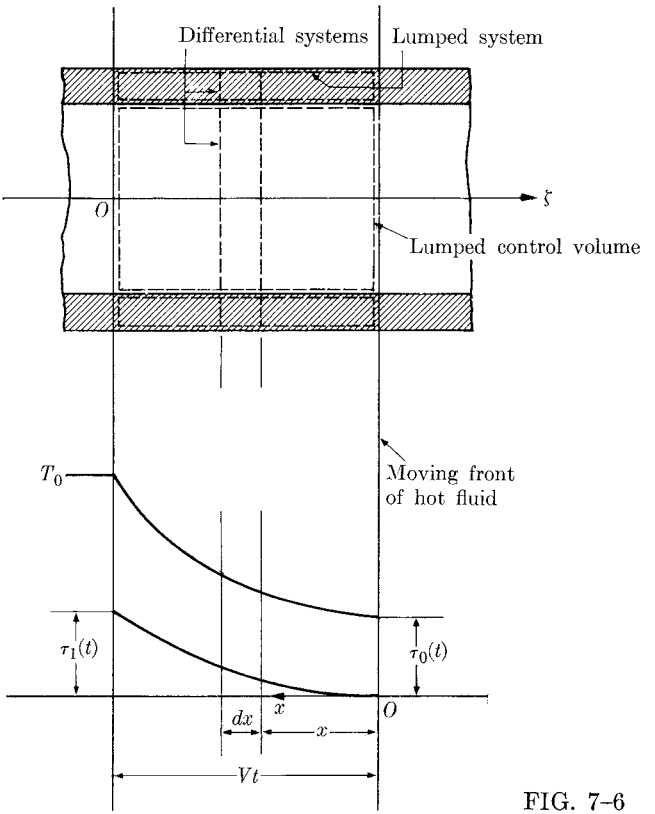


FIG. 7–6

The special case of this problem corresponding to $\epsilon = 1$ may be found in Carslaw and Jaeger† in terms of two integrals, tabulated later by Rizika.‡ $\psi(\xi, \tau^*; \epsilon)$ becomes identical to Rizika's second table provided the space and time variables of this table are replaced by $\epsilon\xi$ and τ^*/ϵ, respectively (Fig. 7–5). (What is the physical significance of $\epsilon = 1$?)

In practice, an approximate solution may be more convenient than the foregoing exact but somewhat complicated solution. Such an approximate solution is obtained in the next example.

Example 7–6. Let us obtain the integral formulation of Example 7–5 and an approximate solution to this formulation.

Consider the lumped control volume and differential system for the fluid, and the lumped and differential systems for the tube walls as shown in Fig. 7–6. Evaluating the terms of the first law of thermodynamics for the lumped control volume and for the lumped system by means of the corresponding differen-

† See Reference 1, p. 394, Eqs. (16) and (17).

‡ See Reference 13, from which our Fig. 7–5 is reproduced by permission.

tial systems, we obtain the first law for the fluid,

$$\frac{d}{dt}\int_0^{Vt} \rho cAT\,dx = \rho cAVT_0 - \int_0^{Vt} h_i P_i(T - \theta)\,dx, \tag{7–88}$$

and for the tube walls,

$$\frac{d}{dt}\int_0^{Vt} \rho' c' A' \theta\,dx = \int_0^{Vt} h_i P_i(T - \theta)\,dx - \int_0^{Vt} h_o P_o \theta\,dx. \tag{7–89}$$

Equations (7–88) and (7–89) may conveniently be rearranged in terms of the constants b_1, b_2, and b_3 as defined in Example 7–5. It follows then that

$$\frac{d}{dt}\int_0^{Vt} T\,dx = VT_0 - b_1\int_0^{Vt} (T - \theta)\,dx, \tag{7–90}$$

$$\frac{d}{dt}\int_0^{Vt} \theta\,dx = b_2\int_0^{Vt} (T - \theta)\,dx - b_3\int_0^{Vt} \theta\,dx. \tag{7–91}$$

Before selecting temperature profiles, let us simplify Eqs. (7–90) and (7–91) in terms of the spacewise averaged values of the fluid and tube-wall temperatures,

$$T_m(t) = \frac{1}{VT_0}\int_0^{Vt} T(x, t)\,dx, \tag{7–92}$$

$$\theta_m(t) = \frac{1}{VT_0}\int_0^{Vt} \theta(x, t)\,dx. \tag{7–93}$$

Inserting Eqs. (7–92) and (7–93) into Eqs. (7–90) and (7–91) gives

$$\frac{dT_m}{dt} = 1 - b_1(T_m - \theta_m), \tag{7–94}$$

$$\frac{d\theta_m}{dt} = b_2(T_m - \theta_m) - b_3\theta_m, \tag{7–95}$$

subject to the initial conditions

$$T_m(0) = 0, \qquad \theta_m(0) = 0, \tag{7–96}$$

which are evident from Eqs. (7–92) and (7–93).

Delaying for the moment the discussion of the spacewise distribution of the temperature profiles, we may proceed here to the solution of the spacewise averaged formulation. Taking the Laplace transforms of Eqs. (7–94) and (7–95), and noting Eq. (7–96), we obtain

$$(p + b_1)\overline{T}_m - b_1\bar{\theta}_m = 1/p, \tag{7–97}$$

$$b_2\overline{T}_m - (p + b_2 + b_3)\bar{\theta}_m = 0. \tag{7–98}$$

Equations (7–97) and (7–98) yield

$$\overline{T}_m(p) = \frac{p + b_2 + b_3}{p[p^2 + (b_1 + b_2 + b_3)p + b_1 b_3]}, \tag{7–99}$$

$$\bar{\theta}_m(p) = \frac{b_2}{p[p^2 + (b_1 + b_2 + b_3)p + b_1 b_3]}, \tag{7–100}$$

which may be rearranged by introducing

$$b_1 + b_2 + b_3 = \lambda_1 + \lambda_2, \qquad b_1 b_3 = \lambda_1 \lambda_2.$$

The result is

$$\overline{T}_m(p) = \frac{1}{(p + \lambda_1)(p + \lambda_2)} + \frac{b_4}{p(p + \lambda_1)(p + \lambda_2)}, \tag{7–101}$$

$$\bar{\theta}_m(p) = \frac{b_2}{p(p + \lambda_1)(p + \lambda_2)}, \tag{7–102}$$

where $b_4 = b_2 + b_3$. Either by first expanding Eqs. (7–101) and (7–102) in partial fractions and then employing TP(12), or by appropriately applying TP(13) and TP(15) to Eqs. (7–101) and (7–102), we obtain

$$T_m(t) = \frac{(b_4/\lambda_2 - 1)(1 - e^{-\lambda_2 t}) - (b_4/\lambda_1 - 1)(1 - e^{-\lambda_1 t})}{\lambda_1 - \lambda_2}, \tag{7–103}$$

$$\theta_m(t) = \frac{(b_2/\lambda_2)(1 - e^{-\lambda_2 t}) - (b_2/\lambda_1)(1 - e^{-\lambda_1 t})}{\lambda_1 - \lambda_2}. \tag{7–104}$$

Finally, let us consider the spacewise distribution of the temperature profiles in terms of the simplest physically admissible polynomials, that is, in terms of parabolas. Referring to Fig. 7–6, and for simplicity of integration selecting as the origin of the coordinate axis x the moving front of the hot fluid, we have the first-order Kantorovich profiles in terms of the temperatures $\tau_0(t)$ and $\tau_1(t)$,

$$T(x, t) = \tau_0(t) + [T_0 - \tau_0(t)](x/Vt)^2, \tag{7–105}$$

$$\theta(x, t) = \tau_1(t)(x/Vt)^2. \tag{7–106}$$

The right-hand sides of Eqs. (7–92) and (7–93) integrated by means of Eqs. (7–105) and (7–106) give the relation between $T_m(t)$ and $\tau_0(t)$, and $\theta_m(t)$ and $\tau_1(t)$,

$$\tau_0(t)/T_0 = \tfrac{3}{2}[T_m(t)/t] - \tfrac{1}{2}, \tag{7–107}$$

$$\tau_1(t)/T_0 = 3\theta_m(t)/t. \tag{7–108}$$

Inserting Eqs. (7–103) and (7–104) into Eqs. (7–107) and (7–108), and the result into Eqs. (7–105) and (7–106), we obtain the approximate temperatures of

the fluid and the tube walls,

$$\frac{T(x,t)}{T_0} = \left(\frac{x}{Vt}\right)^2 + \frac{1}{2}\left\{\left(\frac{3b_4}{\lambda_1 - \lambda_2}\right)\left[\left(1 - \frac{\lambda_2}{b_4}\right)\left(\frac{1 - e^{-\lambda_2 t}}{\lambda_2 t}\right) - \left(1 - \frac{\lambda_1}{b_4}\right)\left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1 t}\right) - 1\right\}\left[1 - \left(\frac{x}{Vt}\right)^2\right]\right. \quad (7\text{–}109)$$

and

$$\frac{\theta(x,t)}{T_0} = \left(\frac{3b_2}{\lambda_1 - \lambda_2}\right)\left[\left(\frac{1 - e^{-\lambda_2 t}}{\lambda_2 t}\right) - \left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1 t}\right)\right]\left(\frac{x}{Vt}\right)^2, \quad (7\text{–}110)$$

both valid when $\zeta \leq Vt$. According to the physics of the problem and Fig. 7–6, the fluid and the tube-wall temperature are both zero when $\zeta > Vt$.

Note the simplicity of the functions involved in Eqs. (7–109) and (7–110), as compared with those involved in the exact solution of the problem, Eqs. (7–86) and (7–87). The detailed comparison of the exact and approximate solutions, although important, requires an involved parametric study, and hence is not considered here.

Example 7–7. Let us reconsider the problem of Examples 7–5 and 7–6, but assume that the outer surface of the tube is insulated. Also, instead of there being a sudden change in the inlet temperature of the fluid, assume that the constant internal energy u''' begins to be generated uniformly in the tube walls (Fig. 7–7). We wish to obtain the radially lumped and axially differential formulation of the problem, and its solution.

The formulation of the problem is

$$\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x} + b_1(T - \theta) = 0, \quad (7\text{–}67)$$

$$\frac{\partial \theta}{\partial t} - b_2(T - \theta) = u, \quad (7\text{–}111)$$

$$T(x, 0) = 0, \quad (7\text{–}69)$$

$$\theta(x, 0) = 0, \quad (7\text{–}70)$$

$$T(0, t) = 0, \quad (7\text{–}112)$$

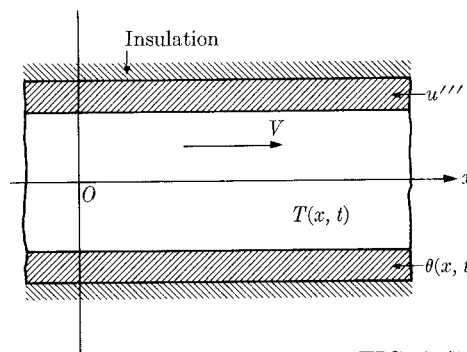


FIG. 7–7

where $u = u'''/\rho' c'$, and b_1 and b_2 are defined as in Example 7–5.

Taking the Laplace transforms of Eqs. (7–67) and (7–111), and employing Eqs. (7–69) and (7–70), we have

$$p\bar{T} + V\frac{d\bar{T}}{dx} + b_1(\bar{T} - \bar{\theta}) = 0, \quad (7\text{–}72)$$

$$p\bar{\theta} - b_2(\bar{T} - \bar{\theta}) = u/p, \quad (7\text{–}113)$$

subject to the transform of Eq. (7–112),

$$\overline{T}(0, p) = 0. \tag{7–114}$$

Solving $\bar{\theta}$ from Eq. (7–113), inserting the result into Eq. (7–72), and rearranging gives

$$\frac{d\overline{T}}{dx} + \frac{1}{V}\left(p + b_1 - \frac{b_1 b_2}{p + b_2}\right)\overline{T} = \frac{ub_1/V}{p(p + b_2)}. \tag{7–115}$$

The general solution of Eq. (7–115) satisfied by Eq. (7–114) is

$$\frac{\overline{T}(x, p)}{u} = \frac{b_1}{p^2(p + b)}\{1 - e^{-b_1 x/V} e^{-(x/V)p} e^{b_1 b_2 x/V(p+b_2)}\}, \tag{7–116}$$

where $b = b_1 + b_2$. Introducing Eq. (7–116) into Eq. (7–113), we obtain

$$\begin{aligned}\frac{\bar{\theta}(x, p)}{u} = {} & \underset{\text{(iv)}}{\left[\frac{1}{p(p + b)} + \frac{b_1}{p^2(p + b)}\right]} \\ & - b_1 b_2 e^{-b_1 x/V} \underset{\text{(i)}}{[e^{-(x/V)p}]} \underset{\text{(iii)}}{\left[\frac{1}{p^2(p + b)}\right]} \underset{\text{(ii)}}{\left[\frac{e^{b_1 b_2 x/V(p+b_2)}}{(p + b_2)}\right]}.\end{aligned} \tag{7–117}$$

For convenience in tabulations and numerical computations, we may follow the steps of Example 7–5 and express the solutions in terms of integrals involving the modified Bessel function of the first kind, of order zero. Equation (7–116) may then be rearranged accordingly by multiplying and dividing the last of its terms by $(p + b_2)$. Thus we have

$$\begin{aligned}\frac{\overline{T}(x, p)}{u} = {} & b_1 \underset{\text{(iii)}}{\left[\frac{1}{p^2(p + b)}\right]} \\ & - b_1 e^{-b_1 x/V} \underset{\text{(i)}}{[e^{-(x/V)p}]} \underset{\text{(v)}}{\left[\frac{1}{p(p + b)} + \frac{b_2}{p^2(p + b)}\right]} \underset{\text{(ii)}}{\left[\frac{e^{b_1 b_2 x/V(p+b_2)}}{(p + b_2)}\right]}.\end{aligned} \tag{7–118}$$

Before the inversion process, let us first note that the terms in the first brackets of Eqs. (7–117) and (7–118) do not involve the factor $e^{-(x/V)p}$, and yield inversions valid for $t \geq 0$. Thus, inserting $\alpha = 0$, $\beta = b$ into TP(13) and TP(16) for (iv) of Eq. (7–117), and again $\alpha = 0$, $\beta = b$ into TP(16) for (iii) of Eq. (7–118), we obtain

$$\frac{T(t)}{ub_1/b^2} = bt - (1 - e^{-bt}), \qquad 0 \leq t \leq x/V, \tag{7–119}$$

$$\frac{\theta(t)}{ub_1/b^2} = bt + \frac{b_2}{b_1}(1 - e^{-bt}), \qquad 0 \leq t \leq x/V. \tag{7–120}$$

[Why are Eqs. (7–119) and (7–120) independent of x? Write the particular formulation of the problem which yields these solutions. What is the physical significance of this formulation?] Introducing the dimensionless quantities $\xi = b_1 x/V$, $\tau = b_2 t$, and $\lambda = 1 + b_1/b_2$, we may rearrange Eqs. (7–119) and (7–120) in the form

$$\frac{T(\tau;\lambda)}{(u'''A'/h_iP_i)(\lambda-1)/\lambda^2} = \lambda\tau - (1 - e^{-\lambda\tau}), \qquad 0 \le \tau \le (\lambda-1)\xi, \tag{7–121}$$

$$\frac{\theta(\tau;\lambda)}{(u'''A'/h_iP_i)(\lambda-1)/\lambda^2} = \lambda\tau + \left(\frac{1}{\lambda-1}\right)(1 - e^{-\lambda\tau}), \qquad 0 \le \tau \le (\lambda-1)\xi. \tag{7–122}$$

When $t \ge x/V$, the solutions given by Eqs. (7–119) and (7–120) must be supplemented by the inversion of the terms involving $e^{-(x/V)p}$. The procedure of Example 7–5 may again be followed. First replacing b_4 by b_2, we have from Eq. (7–80)

$$\bar{g}(p) = \frac{e^{b_1 b_2 x/V(p+b_2)}}{(p+b_2)} = L\left\{e^{-b_2 t} I_0\left[2\left(\frac{b_1 b_2 x t}{V}\right)^{1/2}\right]\right\}. \tag{7–123}$$

Let $\bar{f}_1(p)$ and $\bar{f}_2(p)$ denote (iii) of Eq. (7–117) and (v) of Eq. (7–118), respectively. Noting that the inverse of (iii) has already been found, and the inverse of (v) can be obtained from that of (iv) by simply replacing b_1 of the latter by b_2, we may write

$$f_1(t) = \frac{1}{b^2}[bt + (1 - e^{-bt})], \tag{7–124}$$

$$f_2(t) = \frac{1}{b^2}[bb_2 t - b_1(1 - e^{-bt})]. \tag{7–125}$$

Then the convolution of Eqs. (7–123) and (7–124), and that of Eqs. (7–123) and (7–125) according to TP(VIII), and the use of TP(VII) with $\alpha = x/V$ give the fluid and tube-wall temperatures

$$\begin{aligned}\frac{T(x,t)}{ub_1/b^2} = {} & bt - (1 - e^{-bt}) - e^{-b_1 x/V} \\ & \times \left\{(bb_2 t^* + b_1)\int_0^{t^*} e^{-b_2 s} I_0\left[2\left(\frac{b_1 b_2 x s}{V}\right)^{1/2}\right] ds \right. \\ & - bb_2 \int_0^{t^*} s e^{-b_2 s} I_0\left[2\left(\frac{b_1 b_2 x s}{V}\right)^{1/2}\right] ds \\ & \left. - b_1 e^{-bt^*} \int_0^{t^*} e^{b_1 s} I_0\left[2\left(\frac{b_1 b_2 x s}{V}\right)^{1/2}\right] ds\right\}, \qquad t \ge \frac{x}{V}, \end{aligned} \tag{7–126}$$

$$\frac{\theta(x,t)}{ub_1/b^2} = bt + \frac{b_2}{b_1}(1 - e^{-bt}) - e^{-b_1x/V}$$
$$\times \left\{ b_2(bt^* - 1)\int_0^{t^*} e^{-b_2 s} I_0\left[2\left(\frac{b_1 b_2 xs}{V}\right)^{1/2}\right] ds \right.$$
$$- bb_2\int_0^{t^*} se^{-b_2 s} I_0\left[2\left(\frac{b_1 b_2 xs}{V}\right)^{1/2}\right] ds$$
$$\left. + b_2 e^{-bt^*}\int_0^{t^*} e^{b_1 s} I_0\left[2\left(\frac{b_1 b_2 xs}{V}\right)^{1/2}\right] ds \right\}, \qquad t \geq \frac{x}{V}. \qquad (7\text{–}127)$$

The integrals of Eqs. (7–126) and (7–127), like those of Example 7–5, may be expressed in terms of the series expansion of the appropriate Bessel function, Eq. (7–84). It follows then that

$$\frac{T(\xi,\tau;\lambda)}{(u'''A'/h_iP_i)(\lambda-1)/\lambda^2} = \lambda\tau - (1 - e^{-\lambda\tau})$$
$$- e^{-\xi}[(\lambda\tau^* + \lambda - 1)\Psi(\xi,\tau^*) - \lambda\Phi(\xi,\tau^*) - e^{-\lambda\tau^*}\Lambda(\xi,\tau^*;\lambda)],$$
$$\tau \geq (\lambda - 1)\xi, \qquad (7\text{–}128)$$

$$\frac{\theta(\xi,\tau;\lambda)}{(u'''A'/h_iP_i)(\lambda-1)/\lambda^2} = \lambda\tau + \left(\frac{1}{\lambda-1}\right)(1 - e^{-\lambda\tau})$$
$$- e^{-\xi}\left[(\lambda\tau^* - 1)\Psi(\xi,\tau^*) - \lambda\Phi(\xi,\tau^*) + \left(\frac{1}{\lambda-1}\right)e^{-\lambda\tau^*}\Lambda(\xi,\tau^*;\lambda)\right],$$
$$\tau \geq (\lambda - 1)\xi, \qquad (7\text{–}129)$$

where

$$\Psi(\xi,\tau^*) = \sum_{k=0}^{\infty}\frac{\xi^k}{(k!)^2}\int_0^{\tau^*}\zeta^k e^{-\zeta}\,d\zeta, \qquad (7\text{–}130)$$

$$\Phi(\xi,\tau^*) = \sum_{k=0}^{\infty}\frac{\xi^k}{(k!)^2}\int_0^{\tau^*}\zeta^{k+1} e^{-\zeta}\,d\zeta, \qquad (7\text{–}131)$$

$$\Lambda(\xi,\tau^*;\lambda) = \sum_{k=0}^{\infty}\frac{[\xi/(\lambda-1)]^k}{(k!)^2}\int_0^{(\lambda-1)\tau^*}\zeta^k e^{\zeta}\,d\zeta. \qquad (7\text{–}132)$$

Equation (7–130) is a special case of Eq. (7–85) for $\epsilon = 1$. Equations (7–131) and (7–132) have been tabulated in References 14 and 15;† the results are re-

† The functions Ψ_4 and Ψ_5 of References 14 and 15 correspond to Λ and Φ, respectively. Figures 7–8 and 7–9 reproduced by permission.

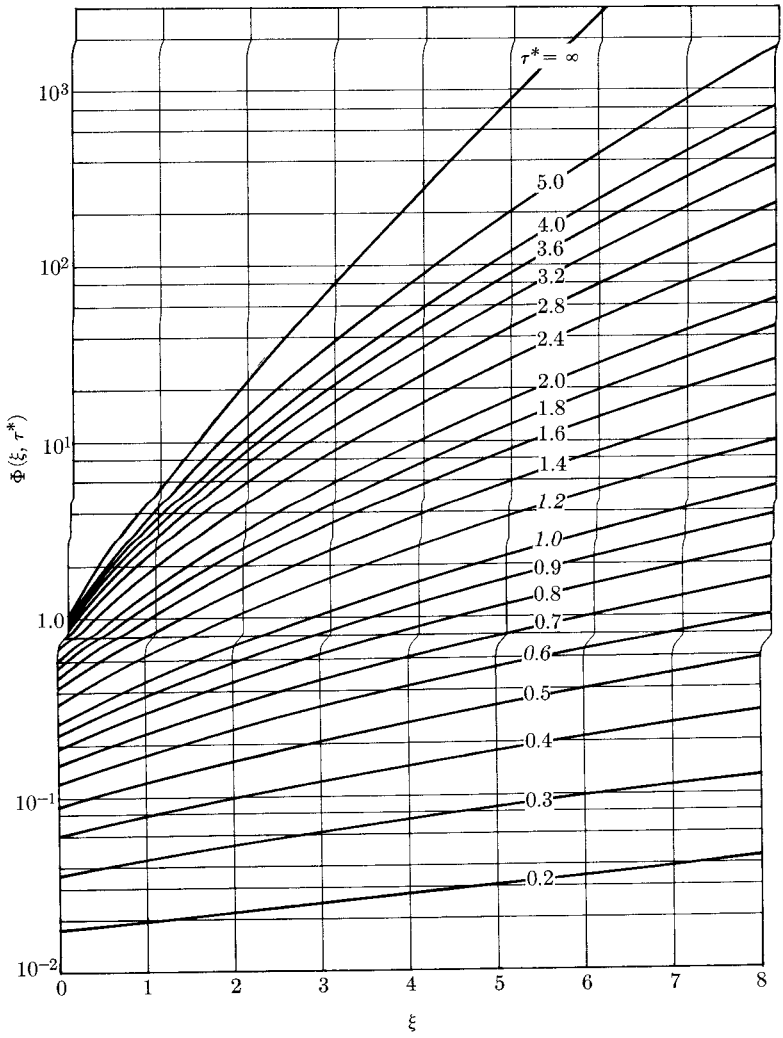


FIG. 7–8

produced here in Figs. 7–8 and 7–9, respectively. A parametric study of the fluid and tube-wall temperatures for various values of λ, although useful, will not be elaborated here.

As we saw in the foregoing examples, the inverse transforms of a large number of problems can be obtained directly by the use of transform tables. There remain, however, many problems whose transforms are not suitable to tables and which thus require the use of the inversion theorem for Laplace transforms. The next three sections are devoted to topics related to this theorem. In Section 7–5, the Fourier integrals and the Fourier transforms are briefly reviewed.

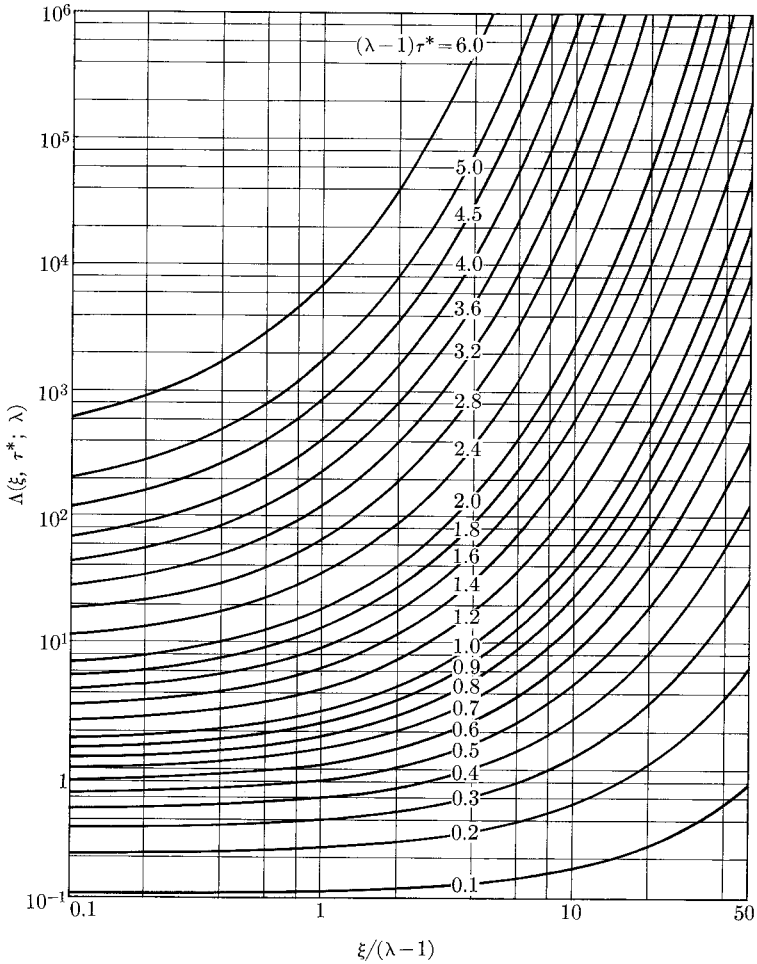


FIG. 7–9

From the definition of Fourier transforms, the inversion theorem is developed in Section 7–6. Then, in Section 7–7, a short treatment is given to functions of complex variables and the theory of residues, which are useful for evaluating the integral of the inversion theorem.

7–5. Fourier Integrals

In Chapters 4 and 5 we solved a number of problems converting them to characteristic-value problems by the technique of separation of variables. In this connection, we showed that linear homogeneous second-order differential equations subject to homogeneous boundary conditions may have an infinite number

of particular solutions, the characteristic functions $\varphi_n(x)$, and that to each of these functions there corresponds a discrete (characteristic) value of a parameter λ_n involved in the differential equation. We then established that any two characteristic functions from a given set are orthogonal with respect to a weighting function $w(x)$ over an interval (a, b), that is,

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)\,dx = 0, \qquad m \neq n. \tag{4–14}$$

For every problem of Chapters 4 and 5, we proceeded by expanding a function $f(x)$, which was related to the nonhomogeneous boundary condition, in terms of a given set of orthogonal functions,

$$f(x) = \sum_{n=0}^{\infty} b_n\varphi_n(x), \tag{4–21}$$

then employed the orthogonality given by Eq. (4–14) to obtain the unknown coefficients

$$b_n = \frac{\int_a^b w(x)f(x)\varphi_n(x)\,dx}{\int_a^b w(x)\varphi_n^2(x)\,dx}. \tag{4–23}$$

We used the method summarized above for problems of finite geometry, and for those of semi-infinite geometry provided the latter were or could be made *homogeneous* in *finite* directions of their geometry. Some problems, however, are *nonhomogeneous* in *infinite* directions of their geometry and require an expansion in terms of an appropriate set of functions over a semi-infinite or infinite interval of a nonhomogeneous direction. In such a case, one or both of the integration limits of Eq. (4–14) extend to infinity, the functions $\varphi_n(x)$ vanish, and Eq. (4–21) fails to represent $f(x)$ over the interval (a, b). It is the purpose of this section to develop a method, that of the so-called Fourier integrals, by which a given function $f(x)$ can be represented in terms of a set of functions over the interval $0 < x < \infty$ or $-\infty < x < \infty$.

Let us first reconsider the problem of the finite interval L,

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0, \tag{4–7}$$

subject to the boundary conditions,

$$y(0) = 0, \tag{4–8}$$

$$y(L) = 0, \tag{4–9}$$

and then see what happens when we let $L \to \infty$. The characteristic functions

corresponding to Eqs. (4–7), (4–8), and (4–9) are $\varphi_n(x) = \sin \lambda_n x$, with the characteristic values $\lambda_n = n\pi/L$, $n = 1, 2, 3, \ldots$ These characteristic functions are orthogonal with respect to the weighting function 1 over the interval $(0, L)$ according to the general proof given by Eqs. (4–16) through (4–20). Now recall from Chapter 4 that for each particular case the orthogonality of the characteristic functions can be shown by direct integration. In the present case, for instance, the use of appropriate trigonometric properties readily gives

$$\frac{2}{L}\int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = \frac{\sin (\lambda_m - \lambda_n)L}{(\lambda_m - \lambda_n)L} - \frac{\sin (\lambda_m + \lambda_n)L}{(\lambda_m + \lambda_n)L}. \tag{7–133}$$

The right-hand side of Eq. (7–133) vanishes if λ_m and λ_n are different integral multiples of π/L, and is equal to unity if λ_m and λ_n both assume the same integral multiple of π/L. When $L \to \infty$, however, the right-hand side approaches zero for any rational values of λ_m and λ_n if $\lambda_m \neq \lambda_n$, but becomes unity if $\lambda_m = \lambda_n$. Hence the limit of Eq. (7–133) becomes

$$\lim_{L\to\infty} \frac{2}{L}\int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = \begin{cases} 0, & \lambda_m \neq \lambda_n, \\ 1, & \lambda_m = \lambda_n. \end{cases} \tag{7–134}$$

Equation (7–134) implies that $\sin \lambda x$, corresponding to any two positive rational but different values of λ, has a sort of orthogonality property in the semi-infinite interval $(0, \infty)$. Hence the representation of a given function $f(x)$ in terms of $\sin \lambda x$ over a semi-infinite interval involves all positive rational values of λ and is not restricted, as in the finite case, to a set of discrete values.† Thus in the limiting case $L \to \infty$, the series representation of an arbitrary function $f(x)$ by the sum of $b_n \sin \lambda_n x$ corresponding to discrete characteristic values should be replaced by the infinite integral of $b(\lambda) \sin \lambda x$ to include all contributions of continuous characteristic values. There then follows

$$f(x) = \int_0^\infty b(\lambda) \sin \lambda x \, d\lambda, \qquad 0 < x < \infty. \tag{7–135}$$

Assuming that such a representation exists, for a sufficiently well-behaved function $f(x)$, we have yet to determine $b(\lambda)$ for all positive values of λ. Following a similar procedure to that employed for finite intervals, that is, multiplying both sides of the Eq. (7–135) by $\sin \lambda^* x$, then integrating the result with respect to x over an interval $(0, L)$, we have

$$\int_0^L f(x) \sin \lambda^* x \, dx = \int_0^L \sin \lambda^* x \left(\int_0^\infty b(\lambda) \sin \lambda x \, d\lambda \right) dx. \tag{7–136}$$

† As with a finite interval, the negative values need not be considered.

Assuming that the order of integration may be interchanged, we may rearrange Eq. (7–136) as

$$\int_0^L f(x) \sin \lambda^* x \, dx = \int_0^\infty b(\lambda) \left(\int_0^L \sin \lambda x \sin \lambda^* x \, dx \right) d\lambda,$$

which by the use of Eq. (7–133) becomes

$$\int_0^L f(x) \sin \lambda^* x \, dx = \frac{L}{2} \int_0^\infty b(\lambda) \left[\frac{\sin (\lambda - \lambda^*)L}{(\lambda - \lambda^*)L} - \frac{\sin (\lambda + \lambda^*)L}{(\lambda + \lambda^*)L} \right] d\lambda. \tag{7–137}$$

Now we introduce the new dummy variables $(\lambda - \lambda^*)L = \beta$ and $(\lambda + \lambda^*)L = \gamma$ for the first and second terms on the right-hand side of Eq. (7–137), respectively, and obtain

$$\int_0^L f(x) \sin \lambda^* x \, dx = \frac{1}{2} \int_{-\lambda^* L}^\infty b\left(\lambda^* + \frac{\beta}{L}\right) \frac{\sin \beta}{\beta} \, d\beta - \frac{1}{2} \int_{\lambda^* L}^\infty b\left(-\lambda^* + \frac{\gamma}{L}\right) \frac{\sin \gamma}{\gamma} \, d\gamma. \tag{7–138}$$

Since $\lambda^* > 0$, when $L \to \infty$ the second integral on the right of Eq. (7–138) approaches zero, whereas the first one yields

$$\frac{1}{2} \int_{-\infty}^\infty b(\lambda^*) \frac{\sin \beta}{\beta} \, d\beta = \frac{b(\lambda^*)}{2} \int_{-\infty}^\infty \frac{\sin \beta}{\beta} \, d\beta. \tag{7–139}$$

The value of the integral on the right of Eq. (7–139) is available in integral tables. It follows then that

$$\int_0^\infty f(x) \sin \lambda^* x \, dx = \frac{\pi}{2} b(\lambda^*), \qquad \lambda^* > 0.$$

To distinguish the dummy variable used here from the independent variable in $f(x)$, x may be replaced by ξ to give

$$b(\lambda) = \frac{2}{\pi} \int_0^\infty f(\xi) \sin \lambda \xi \, d\xi, \tag{7–140}$$

where, for convenience, the parameter λ^* is also replaced by λ.

Inserting Eq. (7–140) into Eq. (7–135), we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \int_0^\infty f(\xi) \sin \lambda \xi \, d\xi \, d\lambda, \qquad 0 < x < \infty. \tag{7–141}$$

Equation (7–141) is the *Fourier sine integral* representation of $f(x)$ and can be shown to be valid when $x > 0$, if $f(x)$ is piecewise differentiable in any positive

interval and if the integral $\int_0^\infty |f(\xi)|\, d\xi$ exists. Like the Fourier series, Eq. (7–141) represents $f(x)$ at points of continuity, and gives $[f(x+) + f(x-)]/2$ at points where $f(x)$ has finite jumps.

The *Fourier cosine integral* representation of $f(x)$ can be obtained in a manner similar to the development of Eq. (7–141). Thus, from the definition of this representation,

$$f(x) = \int_0^\infty a(\lambda) \cos \lambda x \, d\lambda, \tag{7–142}$$

we obtain

$$a(\lambda) = \frac{2}{\pi} \int_0^\infty f(\xi) \cos \lambda \xi \, d\xi. \tag{7–143}$$

Introducing Eq. (7–143) into Eq. (7–142) gives

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(\xi) \cos \lambda \xi \, d\xi \, d\lambda, \qquad 0 < x < \infty. \tag{7–144}$$

As in the finite case, Eq. (7–141) is valid for all values of x when $f(x)$ is an odd function; similarly, Eq. (7–144) is valid for all values of x when $f(x)$ is an even function. Using these facts, we may obtain a representation of an arbitrary function $f(x)$ for all values of x in terms of the sine and cosine integrals. First, writing $f(x)$ in the form†

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)],$$

we get

$$f(x) = f_e(x) + f_o(x),$$

where $f_e(x)$ and $f_o(x)$ are even and odd functions, respectively. It follows then that

$$\begin{aligned} \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^\infty f(\xi) \cos \lambda \xi \, d\xi \, d\lambda \\ = \frac{1}{\pi} \int_0^\infty \cos \lambda x \int_{-\infty}^\infty f_e(\xi) \cos \lambda \xi \, d\xi \, d\lambda \\ = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f_e(\xi) \cos \lambda \xi \, d\xi \, d\lambda = f_e(x), \qquad -\infty < x < \infty, \end{aligned} \tag{7–145}$$

and similarly,

$$\frac{1}{\pi} \int_0^\infty \sin \lambda x \int_{-\infty}^\infty f(\xi) \sin \lambda \xi \, d\xi \, d\lambda = f_o(x), \qquad -\infty < x < \infty. \tag{7–146}$$

† See Fig. 4–1.

Finally, adding Eqs. (7–145) and (7–146), we obtain the representation

$$f(x) = \int_0^\infty [a(\lambda) \cos \lambda x + b(\lambda) \sin \lambda x]\, d\lambda, \qquad -\infty < x < \infty, \tag{7–147}$$

where

$$a(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \lambda \xi \, d\xi, \tag{7–148}$$

$$b(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \sin \lambda \xi \, d\xi. \tag{7–149}$$

Equation (7–147) combined with Eqs. (7–148) and (7–149) becomes

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\int_{-\infty}^{\infty} f(\xi) \cos \lambda(x - \xi)\, d\xi \right] d\lambda, \qquad -\infty < x < \infty. \tag{7–150}$$

Furthermore, noting that its integrand is an even function with respect to λ, we may rearrange Eq. (7–150) in the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) \cos \lambda(x - \xi)\, d\xi \, d\lambda, \qquad -\infty < x < \infty. \tag{7–151}$$

This expression is the *complete Fourier integral* representation of $f(x)$ and represents $f(x)$ for all values of x in the usual sense, if $f(x)$ is piecewise differentiable in every finite interval, and if the integral $\int_{-\infty}^{\infty} |f(\xi)|\, d\xi$ exists. By expressing the cosine in terms of exponentials, we may rearrange Eq. (7–151) in a *complex* form as

$$f(x) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} [f(\xi) e^{i\lambda(x-\xi)} + f(\xi) e^{-i\lambda(x-\xi)}]\, d\xi \, d\lambda.$$

If λ is replaced by $-\lambda$ for the second term of this integrand, then it follows that

$$f(x) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} f(\xi) e^{i\lambda(x-\xi)}\, d\xi \, d\lambda - \frac{1}{2\pi} \int_0^{-\infty} \int_{-\infty}^{\infty} f(\xi) e^{i\lambda(x-\xi)}\, d\xi \, d\lambda.$$

Interchanging the limits of first integration of the second term, we obtain

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \left(\int_{-\infty}^{\infty} f(\xi) e^{-i\lambda \xi}\, d\xi \right) d\lambda, \qquad -\infty < x < \infty. \tag{7–152}$$

Equation (7–152) is known as the *complex Fourier integral* representation of $f(x)$. Denoting the inner integral of Eq. (7–152) by $f(\lambda)$, we have

$$f(\lambda) = \int_{-\infty}^{\infty} f(\xi) e^{-i\lambda \xi}\, d\xi, \qquad -\infty < \lambda < \infty, \tag{7–153}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) e^{i\lambda x}\, d\lambda, \qquad -\infty < x < \infty. \tag{7–154}$$

The function $f(\lambda)$ defined by Eq. (7–153) is called the *Fourier transform* of $f(x)$. For a function $f(x)$ which vanishes when $x < 0$, the Fourier transform becomes identical with the Laplace transform provided $i\lambda$ is replaced by p. Analogous integral representations involving cylindrical (Bessel) functions rather than circular functions can be obtained in a similar manner. The expressions thus obtained are known as the *Fourier-Bessel integral* representations; these will not be considered here.

Before utilizing the Fourier integrals in the development of the inversion theorem for Laplace transforms, let us demonstrate their use in the solution of problems involving a nonhomogeneous direction to be made orthogonal over a semi-infinite domain.

Example 7–8. Let us re-solve Example 7–3 by means of Fourier integrals. The formulation of the problem is

$$\frac{\partial \theta}{\partial t} = a \frac{\partial^2 \theta}{\partial x^2}, \tag{7–43}$$

$$\theta(x, 0) = \theta_0, \tag{7–44}$$

$$\theta(0, t) = 0, \tag{7–45}$$

$$\theta(\infty, t) = \theta_0. \tag{7–46}$$

A product solution of Eq. (7–43) satisfied by Eq. (7–45) has the form

$$\theta_\lambda(x, t) = b(\lambda) e^{-a\lambda^2 t} \sin \lambda x,$$

where λ is the separation parameter to be determined. Since the space interval (the only possible orthogonal direction of the problem) extends to infinity, we seek an integral solution involving all positive values of λ, that is

$$\theta(x, t) = \int_0^\infty b(\lambda) e^{-a\lambda^2 t} \sin \lambda x \, d\lambda, \qquad x > 0. \tag{7–155}$$

Employing Eq. (7–44), we obtain from Eq. (7–155)

$$\theta_0 = \int_0^\infty b(\lambda) \sin \lambda x \, d\lambda,$$

which is the Fourier sine integral representation of θ_0. Here $b(\lambda)$ may be evaluated by means of Eq. (7–140). This gives

$$b(\lambda) = 2 \frac{\theta_0}{\pi} \int_0^\infty \sin \lambda \xi \, d\xi. \tag{7–156}$$

Inserting Eq. (7–156) into Eq. (7–155), we have

$$\frac{\theta(x, t)}{\theta_0} = \frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-a\lambda^2 t} \sin \lambda x \sin \lambda \xi \, d\lambda \, d\xi,$$

which can be rearranged according to Eq. (7–133). The result is

$$\frac{\theta(x,t)}{\theta_0} = \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-a\lambda^2 t}\left[\cos\lambda(x-\xi) - \cos\lambda(x+\xi)\right] d\lambda\, d\xi. \qquad (7\text{–}157)$$

Furthermore, referring to the value of the integral

$$\int_0^\infty e^{-a\lambda^2 t}\cos\lambda x\, d\lambda = \frac{1}{2}\left(\frac{\pi}{at}\right)^{1/2} e^{-x^2/4at},$$

which can be found in any integral tables, we may write Eq. (7–157) in the form

$$\frac{\theta(x,t)}{\theta_0} = \frac{1}{2(\pi at)^{1/2}}\int_0^\infty \left[e^{-(x-\xi)^2/4at} - e^{-(x+\xi)^2/4at}\right] d\xi. \qquad (7\text{–}158)$$

Introducing the variable η^2 for $(x-\xi)^2/4at$ in the first integral of Eq. (7–158) and for $(x+\xi)^2/4at$ in the second integral gives

$$\frac{\theta(x,t)}{\theta_0} = \frac{1}{\pi^{1/2}}\left(\int_{-x/2(at)^{1/2}}^\infty e^{-\eta^2} d\eta - \int_{x/2(at)^{1/2}}^\infty e^{-\eta^2} d\eta\right),$$

or equivalently,†

$$\frac{\theta(x,t)}{\theta_0} = \frac{2}{\pi^{1/2}}\int_0^{x/2(at)^{1/2}} e^{-\eta^2} d\eta = \operatorname{erf}\left[\frac{x}{2(at)^{1/2}}\right]. \qquad (7\text{–}52)$$

Equation (7–52) is identical to the result obtained by means of Laplace transforms.

Comparing the algebra of Examples 7–3 and 7–8 we see that the use of Laplace transforms proved to be more convenient than the use of Fourier integrals, because with Fourier integrals we had to go through the integration process, whereas with Laplace transforms we could refer to the result of integration which was tabulated in advance.

In the next section we develop the inversion theorem for Laplace transforms from Fourier integrals. As we shall see, the use of this theorem, reducing the solution of a given problem to an integration in the complex plane, may not be easier than the use of Fourier integrals.

7–6. Inversion Theorem for Laplace Transforms

Let us consider a function $f(x)$ which vanishes when $x < 0$. According to Eq. (7–152) the complex form of the Fourier integral representation of $e^{-\gamma x}f(x)$‡ is

$$e^{-\gamma x}f(x) = \frac{1}{2\pi}\lim_{L\to\infty}\int_{-L}^{L} e^{i\lambda x}\left(\int_0^\infty e^{-i\lambda\xi}e^{-\gamma\xi}f(\xi)\, d\xi\right) d\lambda, \qquad (7\text{–}159)$$

---

† Note that the problem has been solved without employing the boundary condition at infinity, Eq. (7–46). This condition has already been included in Eq. (7–44), however, and need not be satisfied explicitly.

‡ See the sentence following Eq. (7–160) for the reason for introducing $e^{-\gamma x}$.

where the limit form is used for convenience. [See the integration limits of Eq. (7–161).] Noting that x is a parameter for the right-hand side and multiplying both sides by $e^{\gamma x}$, we may rearrange Eq. (7–159) in the form

$$f(x) = \frac{1}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} e^{(\gamma+i\lambda)x} \left(\int_0^\infty e^{-(\gamma+i\lambda)\xi} f(\xi)\, d\xi \right) d\lambda.$$

Furthermore, introducing the variable $p = \gamma + i\lambda$, we have

$$f(x) = \frac{1}{2\pi i} \lim_{L\to\infty} \int_{\gamma-iL}^{\gamma+iL} e^{px} \left(\int_0^\infty e^{-p\xi} f(\xi)\, d\xi \right) dp. \tag{7–160}$$

The transformation given by Eq. (7–160) can be shown to be legitimate if $f(x)$ is piecewise differentiable and of exponential order,† when γ is sufficiently large that the integral $\int_0^\infty e^{-\gamma\xi} |f(\xi)|\, d\xi$ exists.

By making appropriate changes in the notation, replacing x by t and p by z, and showing the Laplace transform of $f(t)$ by $\bar{f}(p)$,

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt,$$

where p is now complex, we can express $f(t)$ in terms of its transform $\bar{f}(p)$ by the complex integral

$$f(t) = \frac{1}{2\pi i} \lim_{L\to\infty} \int_{\gamma-iL}^{\gamma+iL} e^{tz} \bar{f}(z)\, dz, \tag{7–161}$$

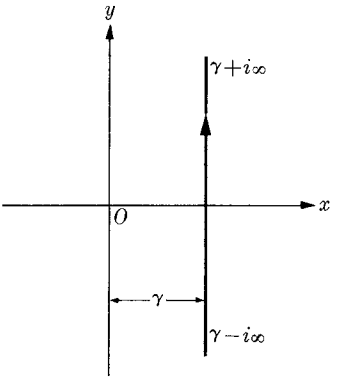


FIG. 7–10

which is known as the *inversion theorem for Laplace transforms.* This integral, as shown in the complex plane of Fig. 7–10, is taken along the infinite line $x = \gamma$, and can usually be evaluated by the contour integration to be explained in Section 7–8.

The inversion theorem given by Eq. (7–161) is also valid for multidimensional problems. If the Laplace transform of the function $f(\mathbf{r}, t)$ with respect to time is $\bar{f}(\mathbf{r}, p)$, then the inversion theorem obtained by rearranging Eq. (7–161) has the form

$$f(\mathbf{r}, t) = \frac{1}{2\pi i} \lim_{L\to\infty} \int_{\gamma-iL}^{\gamma+iL} e^{tz} \bar{f}(\mathbf{r}, z)\, dz, \tag{7–162}$$

where $\mathbf{r}$ denotes the position (space) vector. The inversion theorem may also be used for multidimensional problems to be (Laplace) transformed in a suitable space direction.

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† Note the existence conditions of Eq. (7–4).

In the next section we review the functions of a complex variable as preparation for the application of the inversion theorem.

7–7. Functions of a Complex Variable

A complete treatment of the functions of a complex variable is beyond the scope of the text. Only the definitions and theorems that are needed in the evaluation of the integral appearing in the inversion theorem are given below, and in most cases we indicate the proofs briefly without careful statement of the conditions. For a more extensive study, the reader may refer to any text on the theory of functions of a complex variable.

(1) *Complex variables.* The complex variable z is an expression of the form

$$z = x + iy, \tag{7–163}$$

where x and y are real variables (the so-called *real* and *imaginary* parts of z), and i is the *imaginary unit* defined by $i^2 = -1$.

It is convenient to represent z by the point $P(x, y)$ in the cartesian coordinate system known as the *complex plane* (Fig. 7–11). By definition, the real and imaginary numbers constitute the x- and y-axes, respectively, of this plane. Then a complex number may also be considered as a vector from the origin to the point $P(x, y)$. The length of this vector is the *absolute value* or the *modulus* of z and is denoted by $|z|$. Thus

$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

From Fig. 7–11 it also follows that

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta. \tag{7–164}$$

Combining Eqs. (7–163) and (7–164), we obtain the polar form of z as

$$z = \rho(\cos \theta + i \sin \theta), \tag{7–165}$$

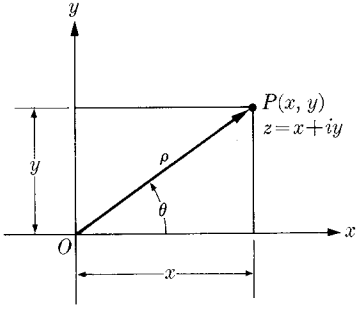


FIG. 7–11

where ρ is the modulus and θ is the so-called *argument* of z. Note that θ is not unique, and it can only be determined within an integral multiple of 2π.

Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

The *conjugate* of z is denoted by $\bar{z}$, and is defined to be

$$\bar{z} = x - iy.$$

The rules for adding, subtracting, multiplying, and dividing complex numbers are the same as those for real numbers.

(2) *Elementary functions.* The elementary functions of a complex variable, such as *polynomials, power series, exponential functions,* and *trigonometric* (circular and hyperbolic) *functions,* are so defined that they reduce to those of real variables when z is replaced by a real variable. The *complex logarithmic function* $\operatorname{Ln} z$ and the *generalized power function* z^a, where a may be real or complex, will be defined later, in (6).

From the definitions of $\sin z$ and $\cos z$ we obtain

$$e^{iz} = \cos z + i \sin z, \tag{7–166}$$

which is known as *Euler's formula.* The polar form of z can be rearranged by the use of Eq. (7–166). Thus, by combining Eqs. (7–165) and (7–166), we have the form

$$z = \rho e^{i\theta}, \tag{7–167}$$

which will prove useful in our later discussions.

Comparing the definitions of the circular and hyperbolic functions, we find the following relations:

$$\begin{aligned} \sinh iz &= i \sin z, & \cosh iz &= \cos z, \\ \sin iz &= i \sinh z, & \cos iz &= \cosh z. \end{aligned} \tag{7–168}$$

(3) *Single-valued functions.* If a function $f(z)$ and the complex variable z are so related that to each value of z in a given region R of the complex plane there corresponds one value of $f(z)$, then $f(z)$ is said to be a *single-valued* function of z. On the other hand, if more than one value of $f(z)$ corresponds to each value of z, then $f(z)$ is *multiple-valued.* Polynomials, power series, the exponential functions, and the trigonometric functions are single-valued functions. The complex logarithmic function and the generalized power function, as we shall see in (6), are multiple-valued functions.

(4) *Existence of* $df(z)/dz$. The first derivative of $f(z)$ at a point exists if the limit

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{7–169}$$

is finite and unique as $\Delta z \to 0$ from any direction in the complex plane.

Suppose that the limit given by Eq. (7–169) exists. Then denoting the real and imaginary parts of $f(z)$ by u and v, respectively, we have

$$\frac{df}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta u + i\, \Delta v}{\Delta x + i\, \Delta y} \right). \tag{7–170}$$

Since Eq. (7–170) must exist regardless of the direction in which $\Delta z \to 0$, let

us first approach this limit along a line parallel to the x-axis; thus

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \left(\frac{\Delta u + i\,\Delta v}{\Delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}. \tag{7–171}$$

Similarly, an approach parallel to the y-axis gives

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \left(\frac{\Delta u + i\,\Delta v}{i\,\Delta y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \tag{7–172}$$

The assumed existence (finiteness and uniqueness) of df/dz requires the equality of Eqs. (7–171) and (7–172); this equality obtains when the respective real and imaginary parts of these equations are equal. Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}. \tag{7–173}$$

These equations are known as the *Cauchy-Riemann conditions.* Thus *the derivative df/dz of a single-valued function $f(z) = u + iv$ exists in a region R of the complex plane if and only if the Cauchy-Riemann conditions are satisfied in R.*

(5) *Analytic functions.* A function $f(z)$ is said to be *analytic* in a region R if $f(z)$ is single-valued in R and its derivative df/dz exists at each point of R.

(6) *Singularities of analytic functions.* The points at which $f(z)$ is not analytic are called the *singular points* of $f(z)$. Then from the definition of an analytic function, the point $z = a$ is a singular point of $f(z)$ if (i) $f(a)$ is not single-valued, or (ii) $df(a)/dz$ does not exist.

(i) The points at which $f(z)$ is not single-valued are called the *branch points* of $f(z)$. When a point $P(z)$ traces out a closed path around $z = a$ without enclosing any other singular point, and the value assumed by the function $f(z)$ differs from its initial value, then, by definition, $z = a$ is a branch point of $f(z)$. A function having branch points is a *multiple-valued* function; each particular value is called a *branch* of this function. A function having branch points in a region R can be made analytic in this region by excluding these points from R by proper *cuts.* Let us illustrate the multiple-valued functions by considering two elementary functions.

The *complex logarithmic function,* represented temporarily by the notation

$$w = \operatorname{Ln} z, \tag{7–174}$$

is defined as the inverse of the exponential function

$$z = e^w. \tag{7–175}$$

Denoting the real and imaginary parts of Eq. (7–174) by u and v, respectively,

and employing Eq. (7–166), we may rearrange Eq. (7–175) as

$$z = x + iy = e^{u+iv} = e^u(\cos v + i \sin v). \tag{7–176}$$

It follows from the real and imaginary parts of Eq. (7–176) that

$$x = e^u \cos v, \qquad y = e^u \sin v. \tag{7–177}$$

Then solving Eq. (7–177) for u and v, we obtain

$$e^{2u} = x^2 + y^2 = |z|^2 = \rho^2, \tag{7–178}$$

$$\tan v = x/y = \tan \theta. \tag{7–179}$$

Since Eq. (7–178) is in terms of real variables, the ordinary *real logarithm* of this equation gives

$$u = \ln |z| = \ln \rho. \tag{7–180}$$

Furthermore, we have from Eq. (7–179)

$$v = \theta \mp 2k\pi, \qquad k = 0, 1, 2, \ldots, \tag{7–181}$$

where θ lies in the range $0 \leq \theta < 2\pi$. Inserting Eqs. (7–180) and (7–181) into Eq. (7–174), we get the complex logarithmic function

$$\operatorname{Ln} z = \ln |z| + i(\theta \mp 2k\pi), \qquad k = 0, 1, 2, \ldots \tag{7–182}$$

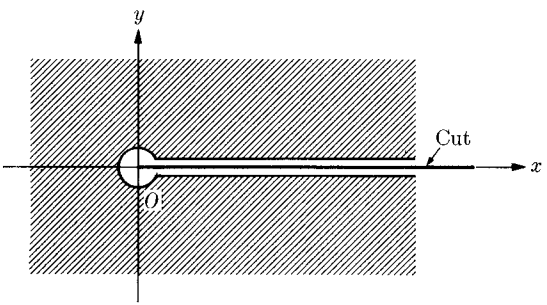


FIG. 7–12

Equation (7–182) clearly shows that $\operatorname{Ln} z$ is a single-valued function for all values of z except the branch point $z = 0$, where it becomes *infinitely many-valued.* To prevent the transition from one branch to another we may introduce a cut, say along the entire positive real axis (Fig. 7–12), thus making $\operatorname{Ln} z$ single-valued on the *cut complex plane.* For convenience in a given case, the cut of Fig. 7–12 can be replaced by a cut along any ray or curve extending from $z = 0$ to infinity.

The generalized power function z^a, where a may be real or complex, is defined in terms of the complex logarithm by the relation

$$z^a = e^{a \operatorname{Ln} z}. \tag{7–183}$$

If z is real and positive and a is real, Eq. (7–183) gives the usual definition of the real logarithm. If z is complex and a is a positive integer, say $a = n$, then inserting Eq. (7–182) into Eq. (7–183) and noting that $|z| = \rho$, we have

$$z^n = e^{n[\ln\rho + i(\theta \mp 2k\pi)]}, \qquad k = 0, 1, 2, \ldots, \tag{7–184}$$

which may be rearranged as

$$z^n = e^{\ln\rho^n} e^{in\theta} e^{\mp 2ink\pi}. \tag{7–185}$$

Since both n and k are integers, Eq. (7–166) gives

$$e^{\mp 2ink\pi} = \cos 2nk\pi \mp i \sin 2nk\pi = 1.$$

Then Eq. (7–185) becomes

$$z^n = \rho^n e^{in\theta}, \qquad n \text{ an integer.} \tag{7–186}$$

Inspection of Eq. (7–186) reveals that z^n is a single-valued function. More generally, if a is a real rational number,† say $a = n/m$ and n, m are integers having no common factor, then Eq. (7–183) takes the form

$$z^{n/m} = \rho^{n/m} e^{i(n/m)\theta} e^{\mp 2i(n/m)k\pi}, \qquad k = 0, 1, 2, \ldots \tag{7–187}$$

Since n and m are given integers, whereas k assumes all integer values, the factor $e^{\mp 2i(n/m)k\pi}$ takes on m different values for each sign of the exponent when $k < m$, but $2m$ values so obtained are repeated periodically when $k \geq m$. Hence $z^{n/m}$ is exactly a *2m-valued function*. Since $z = 0$ is again a branch point, the cut shown in Fig. 7–12 for Ln z may also be used for $z^{n/m}$.

(ii) Let us consider next the singularity of a function $f(z)$ at a point $z = a$ arising from the nonexistence of $df(a)/dz$. In this case if $(z - a)^k f(z)$ is analytic at $z = a$, where k is a positive integer, then $z = a$ is said to be a *pole* of $f(z)$. The smallest value of k which satisfies this condition is the *order* of the pole. For example, $f(z) = 1/(z - a)^k$, where k is a positive integer, has a pole of order k at $z = a$. Similarly the kth term of $f(z) = e^{1/z} = \sum_{k=0}^{\infty} (1/k!)(1/z^k)$ has a pole of order k at $z = 0$.

(7) *The complex line integral. The Cauchy integral theorem.* Let C be any continuous curve of finite length joining the points P_1 and P_2, and let $f(z) = u + iv$ be a continuous function defined at all points of C (Fig. 7–13). The line integral of $f(z)$ along C is defined by

$$\begin{aligned} \int_C f(z)\, dz &= \int_C (u + iv)(dx + i\, dy) \\ &= \int_C (u\, dx - v\, dy) + i \int_C (v\, dx + u\, dy). \end{aligned} \tag{7–188}$$

† The most general case, corresponding to a complex a, is not considered here.

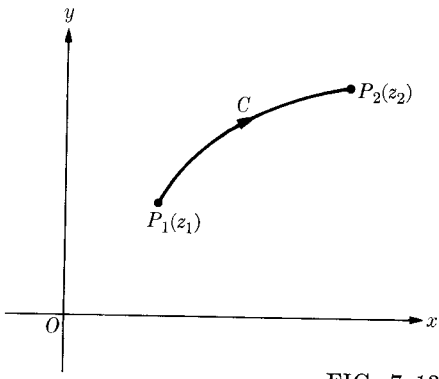


FIG. 7–13

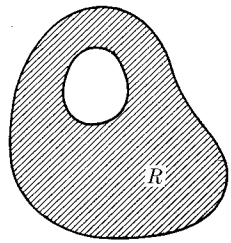


FIG. 7–14

The real and imaginary parts of Eq. (7–188) are ordinary real line integrals. If the integrands of these integrals are exact differentials, say dF and dG, then

$$\begin{aligned} dF &= u\,dx - v\,dy = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy, \\ dG &= v\,dx + u\,dy = \frac{\partial G}{\partial x}\,dx + \frac{\partial G}{\partial y}\,dy. \end{aligned} \tag{7–189}$$

Furthermore, if the partial derivatives involved in Eq. (7–189) are continuous, then the order of differentiation is irrelevant, and we obtain

$$\frac{\partial^2 F}{\partial x\,\partial y}\left(\text{or } -\frac{\partial v}{\partial x}\right) = \frac{\partial^2 F}{\partial y\,\partial x}\left(\text{or } \frac{\partial u}{\partial y}\right),$$

$$\frac{\partial^2 G}{\partial x\,\partial y}\left(\text{or } \frac{\partial u}{\partial x}\right) = \frac{\partial^2 G}{\partial y\,\partial x}\left(\text{or } \frac{\partial v}{\partial y}\right),$$

which are precisely the Cauchy-Riemann conditions. It follows that the *line integral* $\int_C f(z)\,dz$ *is independent of the path* C *joining the end points* P_1 *and* P_2 *if* C *can be enclosed in a simply connected region* R *in which* $f(z)$ *is analytic.*

A *simply connected region* R is so defined that any closed curve lying in R can be continuously shrunk to a point without crossing the boundaries of R. If the curve cannot be so shrunk, we have a *multiply connected region* as demonstrated by Fig. 7–14.

Denoting then $f(z)\,dz$ by an exact differential function $F(z)$, we can evaluate the *line integral of the analytic function* $f(z)$ in the usual manner, according to

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

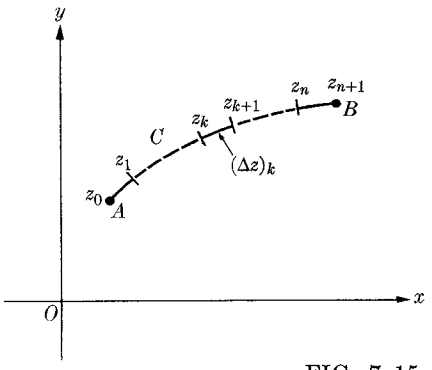


FIG. 7–15

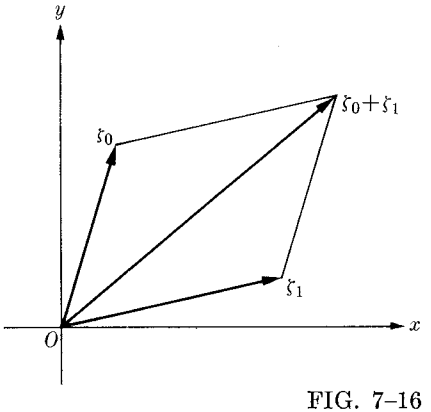


FIG. 7–16

Particularly, if the end points coincide, we have

$$\oint_C f(z)\,dz = 0. \tag{7–190}$$

This result is known as *Cauchy's integral theorem.*

(8) *An upper bound for complex line integrals.* By definition, the complex line integral of an analytic function $f(z)$ along a path C (Fig. 7–15) is the limit of the sum

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=0}^{n} f(z_k)(\Delta z)_k. \tag{7–191}$$

On the other hand, remembering that a complex number can be represented by a vector on the complex plane, we have from Fig. 7–16

$$|\zeta_0 + \zeta_1| \leq |\zeta_0| + |\zeta_1|,$$

which can be readily generalized to the sum of n complex numbers as

$$\left|\sum_{k=0}^{n} \zeta_k\right| \leq \sum_{k=0}^{n} |\zeta_k|. \tag{7–192}$$

Employing Eq. (7–192) with the consideration that $\zeta_k = f(z_k)(\Delta z)_k$, we may rearrange the absolute value of the right-hand side of Eq. (7–191) to give

$$\left|\sum_{k=0}^{n} f(z_k)(\Delta z)_k\right| \leq \sum_{k=0}^{n} |f(z_k)(\Delta z)_k| = \sum_{k=0}^{n} |f(z_k)|\,|(\Delta z)_k|. \tag{7–193}$$

Then the absolute value of Eq. (7–191) becomes

$$\left|\int_C f(z)\,dz\right| \leq \lim_{n\to\infty} \sum_{k=0}^{n} |f(z_k)|\,|(\Delta z)_k|. \tag{7–194}$$

Proceeding to the limit for the right-hand side of Eq. (7–194) as $n \to \infty$, we obtain

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz|. \qquad (7\text{–}195)$$

Furthermore, noting that $|dz| = ds$ is the differential arc length along C, we may write Eq. (7–195) in the form

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,ds. \qquad (7\text{–}196)$$

If $|f(z)|$ is bounded, say $|f(z)| \leq M$ on C, then Eq. (7–196) yields

$$\left|\int_C f(z)\,dz\right| \leq ML, \qquad (7\text{–}197)$$

where L is the length of path C between the points A and B.

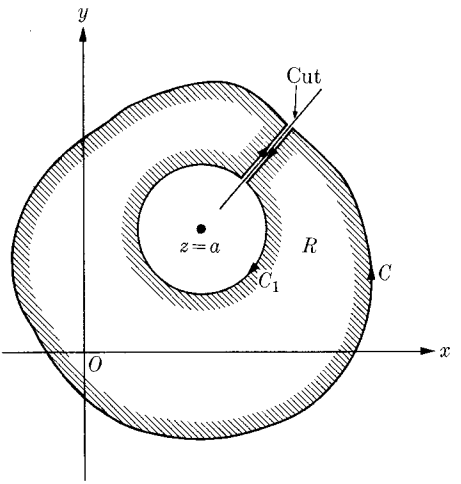


FIG. 7–17

(9) *The value of* $\int_C (z - a)^n\,dz$ *for integer values of* n. If n is zero or a positive integer, then according to the discussion on the generalized power function,† $(z - a)^n$ is analytic everywhere in the complex plane, and Cauchy's integral theorem, Eq. (7–190), gives

$$\oint_C (z - a)^n\,dz = 0, \qquad n \text{ zero or a positive integer}, \qquad (7\text{–}198)$$

where the region enclosed by the contour C may or may not include the point $z = a$. If n is a negative integer, then $(z - a)^n$, having a pole of order n, is not analytic at the point $z = a$. For a region excluding $z = a$, $(z - a)^n$ is analytic and Eq. (7–198) still holds; however, when the region includes $z = a$, $(z - a)^n$ is no longer analytic and Eq. (7–198) ceases to be valid. In this case we proceed as follows. Let C_1 be the unit circle around $z = a$ and let C be an arbitrary contour enclosing C_1. The region R between C and C_1 is not simply connected but can be made so by introducing an arbitrary cut as shown in Fig. 7–17. Since $(z - a)^n$ is analytic in the cut R region, employing Cauchy's integral theorem, we obtain

$$\oint_C (z - a)^n\,dz - \oint_{C_1} (z - a)^n\,dz = 0, \qquad (7\text{–}199)$$

where the first and second integrals are taken in the positive and negative directions, respectively.‡ The integrals along the cut, being considered in oppo-

† See Eqs. (7–183) through (7–187).

‡ Hereafter the counterclockwise and clockwise directions are assumed to be the positive and negative directions, respectively.

site directions, cancel as the width of the cut approaches zero. The first integral of Eq. (7–199) may now be evaluated in terms of the second integral. Noting that $\rho = 1$, we find that the polar expression of C_1 is $z = a + e^{i\theta}$. Then $dz = ie^{i\theta}\, d\theta$, and

$$\oint_{C_1} (z - a)^n\, dz = i\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$= i\int_0^{2\pi} [\cos (n + 1)\theta + i \sin (n + 1)\theta]\, d\theta. \qquad (7\text{–}200)$$

Thus, by direct integration of Eq. (7–200), we find that

$$\oint_{C_1} (z - a)^n\, dz = \begin{cases} 0, & n \neq -1, \text{ but a negative integer,} \\ 2\pi i, & n = -1. \end{cases} \qquad (7\text{–}201)$$

Inserting Eq. (7–201) into Eq. (7–199), and considering Eq. (7–198), we obtain the important relation†

$$\oint_C (z - a)^n\, dz = \begin{cases} 0, & n \neq -1, \text{ but any other integer or zero,} \\ 2\pi i, & n = -1, \end{cases} \qquad (7\text{–}202)$$

where C is an arbitrary contour enclosing the point $z = a$.

(10) *The Cauchy integral formula.* Let $f(z)$ be an analytic function in a simply connected region R bounded by a contour C, and let a be a point inside C such that $f(a) \neq 0$. Then the function

$$f(z)/(z - a) \qquad (7\text{–}203)$$

is analytic at all points of R except $z = a$, where it has a simple pole. Let C_ρ be a small circle of radius ρ with center at $z = a$. Now the function defined by Eq. (7–203) is analytic between the contours C and C_ρ (Fig. 7–18), and according to Cauchy's integral theorem we have

$$\oint_C \frac{f(z)\, dz}{(z - a)} = \oint_{C_\rho} \frac{f(z)\, dz}{(z - a)}. \qquad (7\text{–}204)$$

Employing the polar representation $z = a + \rho e^{i\theta}$ of C_ρ and noting that $dz = i\rho e^{i\theta}\, d\theta$, we may rearrange the right-hand side of Eq. (7–204) as follows:

$$\oint_{C_\rho} \frac{f(z)\, dz}{(z - a)} = i\int_0^{2\pi} f(a + \rho e^{i\theta})\, d\theta. \qquad (7\text{–}205)$$

Then the combination of Eq. (7–204) with the limiting value of Eq. (7–205) as

† Note the conceptual similarity between Eq. (7–202) and the condition of orthogonality over a finite or infinite interval.

$\rho \to 0$ gives

$$\oint_C \frac{f(z)\,dz}{(z-a)} = 2\pi i f(a). \tag{7–206}$$

Equation (7–206) is known as *Cauchy's integral formula;* it gives the value of a function $f(z)$ at any point a within a contour C in terms of an integral taken along C in which $f(z)$ is analytic.

Assuming that differentiation under the integral sign is permissible, and differentiating both sides of Eq. (7–206) with respect to a, we obtain

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2}.$$

In a similar manner it can be shown that

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}. \tag{7–207}$$

We therefore conclude that *if $f(z)$ is analytic at a point $z = a$, then derivatives of $f(z)$ of all orders exist at that point.*†

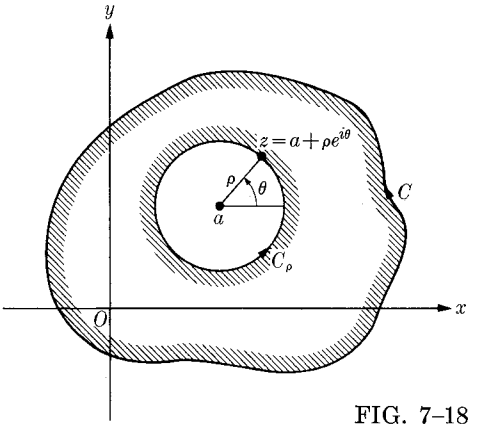


FIG. 7–18

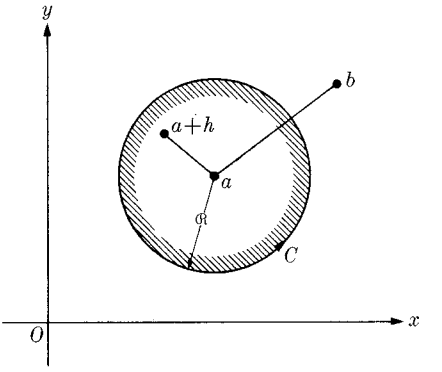


FIG. 7–19

(11) *The Taylor series.* Let $z = a$ be any point at which $f(z)$ is analytic, and let $z = b$ be the nearest singularity of $f(z)$ to $z = a$. Then $f(z)$ is analytic at all points on and in the circle C of radius $\Re < |b - a|$ and with center at $z = a$ (Fig. 7–19). If $z = a + h$ is a point within C, the value of $f(z)$ at this point can be evaluated by Cauchy's integral formula, Eq. (7–206). Thus we have

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)}. \tag{7–208}$$

† Since no branch point is assumed, all derivatives are single-valued. Therefore, if a derivative exists, then the previous derivative is analytic. This implies that the derivatives of all orders are analytic.

Noting that by algebraic manipulation

$$\frac{1}{(z - a - h)} = \frac{1}{(z - a)[1 - h/(z - a)]} = \sum_{n=0}^{\infty} \frac{h^n}{(z - a)^{n+1}}, \quad (7\text{–}209)$$

and inserting Eq. (7–209) into Eq. (7–208), we obtain

$$f(a + h) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left[\oint_C \frac{f(z)\, dz}{(z - a)^{n+1}} \right] h^n. \quad (7\text{–}210)$$

Finally, combination of Eqs. (7–207) and (7–210) gives

$$f(a + h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} h^n. \quad (7\text{–}211)$$

Equation (7–211) is identical in form to the *Taylor series* for functions of a real variable. The transformation $z = a + h$ gives a more frequently used representation of this series as follows:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n. \quad (7\text{–}212)$$

Thus *if $f(z)$ is analytic at a point $z = a$, then derivatives of $f(z)$ of all orders exist, and $f(z)$ can be expanded into a Taylor series; the circle of convergence of this series is the largest circle (with center at $z = a$) in which $f(z)$ is analytic.*

(12) *The Cauchy residue theorem.* Suppose that $z = a$ is a pole of order k and is the only singularity of a function $f(z)$ in a region R. Then, according to the definition of the pole, $(z - a)^k f(z)$ is analytic and, by use of Eq. (7–212), can be expanded into a Taylor series

$$(z - a)^k f(z) = \sum_{n=0}^{\infty} A_n (z - a)^n, \quad (7\text{–}213)$$

where

$$A_n = \frac{1}{n!} \left\{ \frac{d^n}{dz^n} [(z - a)^k f(z)] \right\}_{z=a}. \quad (7\text{–}214)$$

If $z \neq a$, then dividing both sides of Eq. (7–213) by $(z - a)^k$, we get

$$f(z) = \sum_{n=0}^{\infty} A_n (z - a)^{n-k}. \quad (7\text{–}215)$$

Integration of Eq. (7–215) around a contour C enclosing $z = a$ gives

$$\oint_C f(z)\, dz = \sum_{n=0}^{\infty} A_n \oint_C (z - a)^{n-k}\, dz. \quad (7\text{–}216)$$

However, according to Eq. (7–202) we have for any integer value of $(n - k)$

$$\oint_C (z - a)^{n-k}\, dz = \begin{cases} 0, & n - k \neq -1, \\ 2\pi i, & n - k = -1. \end{cases} \tag{7–217}$$

Finally, inserting Eq. (7–217) into the right-hand side of Eq. (7–216), we obtain

$$\oint_C f(z)\, dz = 2\pi i A_{k-1}. \tag{7–218}$$

Equation (7–218) is known as *Cauchy's residue theorem;* the coefficient A_{k-1} is called the *residue* of $f(z)$ at $z = a$, and is usually denoted by Res (a) or, more explicitly, by Res $[f(z); a]$. Hence, if $f(z)$ has a pole of order k at $z = a$, then

$$\oint_C f(z)\, dz = 2\pi i \text{ Res } (a), \tag{7–219}$$

where

$$\text{Res } (a) = \frac{1}{(k-1)!} \left\{ \frac{d^{k-1}}{dz^{k-1}} [(z - a)^k f(z)] \right\}_{z=a}, \tag{7–220}$$

and C is a contour surrounding $z = a$ but excluding all other singularities of $f(z)$. It is clear from Eq. (7–215) that Res (a) is the coefficient of $1/(z - a)$ in the expansion of $f(z)$ in powers of $(z - a)$, and can be either calculated from Eq. (7–220) or obtained directly from the expansion of $f(z)$. The latter proves useful when the order of the pole cannot be readily determined.

The special case of Eq. (7–220) that corresponds to a *simple* pole $(k = 1)$ at $z = a$ gives

$$\text{Res } (a) = \lim_{z \to a} [(z - a) f(z)], \tag{7–221}$$

where the use of the limit becomes necessary, since $f(z) \to \infty$ and the product $(z - a)f(z)$ takes the indeterminate form $0 \times \infty$ as $z \to a$.

Particularly, if $f(z)$ takes the frequently encountered form

$$f(z) = \frac{P(z)}{z^k Q(z)}, \tag{7–222}$$

with a_n, $n = 1, 2, \ldots, N$ as the *simple* roots of $Q(z)$, and $Q(0) \neq 0$, $P(0) \neq 0$, and $P(a_n) \neq 0$, then for the residues at $z = a_n$, we have from Eq. (7–221)

$$\sum_{n=1}^{N} \text{Res } (a_n) = \sum_{n=1}^{N} \lim_{z \to a_n} \frac{(z - a_n) P(z)}{z^k Q(z)}. \tag{7–223}$$

Now L'Hôpital's rule applied to Eq. (7–223) yields

$$\sum_{n=1}^{N} \text{Res } (a_n) = \sum_{n=1}^{N} \lim_{z \to a_n} \frac{P(z) + (z - a_n)(dP/dz)}{kz^{k-1} Q(z) + z^k (dQ/dz)}. \tag{7–224}$$

It follows from Eq. (7–224) that

$$\sum_{n=1}^{N} \text{Res}\,(a_n) = \sum_{n=1}^{N} \frac{P(a_n)}{a_n^k (dQ/dz)_{z=a_n}}. \tag{7–225}$$

The evaluation of the residue for the multiple pole at $z = 0$ may be obtained either from the coefficient of $1/z$ in the expansion of $f(z)$ or by employing Eq. (7–220).

As another particular case, if $f(z)$ has the form

$$f(z) = \frac{P(z)}{Q(z)R(z)}, \tag{7–226}$$

with a_n, $n = 1, 2, \ldots, N$ and b_m, $m = 1, 2, \ldots, M$ as the *simple* roots of $Q(z)$ and $R(z)$, respectively, and $a_n \neq b_m$, $P(a_n) \neq 0$, and $P(b_m) \neq 0$, then following a procedure similar to that of the previous particular case, we may easily show the sum of the residues at all the poles to be

$$\sum_{n=1}^{N} \text{Res}\,(a_n) + \sum_{m=1}^{M} \text{Res}\,(b_m) = \sum_{n=1}^{N} \frac{P(a_n)}{(dQ/dz)_{z=a_n} R(a_n)} + \sum_{m=1}^{M} \frac{P(b_m)}{Q(b_m)(dR/dz)_{z=b_m}}. \tag{7–227}$$

This completes our review of the functions of a complex variable. In the next section we express the inversion integral given by Eq. (7–161) or by Eq. (7–162) in terms of two particular contours which are suitable to our problems.

7–8. Evaluation of the Inversion Theorem in Terms of Two Particular Contours

Let us return now to the solution of conduction problems by Laplace transforms. Suppose that the solution of a given problem has been reduced to the inversion of its transform, $\overline{T}(\mathbf{r}, p)$. If $\overline{T}(\mathbf{r}, p)$ does not appear in or cannot be made suitable to the direct use of tables of transforms, the inversion theorem given by Eq. (7–162) may be employed to write the solution $T(\mathbf{r}, t)$ in terms of a contour integral in the complex plane. Two standard procedures are available for evaluating this contour integral.

(i) *If $\overline{T}(\mathbf{r}, z)$ is analytic except for a number (say N) of poles,* all of which are to the left of some line $x = \gamma$, we complete the contour of Eq. (7–162) and Fig. 7–20(a) by a large circle C of radius $\mathcal{R}$, enclosing all poles of the integrand (Fig. 7–20b). (Hereafter we shall refer to the closed contour of Fig. 7–20(b) as Contour I.) In the limit as $N \to \infty$, it follows from Cauchy's residue theo-

rem, Eq. (7–219), that

$$\lim_{L\to\infty}\int_{\gamma-iL}^{\gamma+iL} e^{tz}\overline{T}(\mathbf{r}, z)\,dz + \lim_{\Re\to\infty}\int_{C} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = 2\pi i \lim_{N\to\infty}\sum_{n=1}^{N} \text{Res}\,(a_n). \tag{7–228}$$

In all the problems we discuss here it can be shown by employing Eq. (7–197) that the integral over the circle C vanishes as $\Re \to \infty$. Thus, in the limit, we have

$$T(\mathbf{r}, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = \sum_{n=1}^{\infty} \text{Res}\,(a_n). \tag{7–229}$$

This case usually arises in problems involving conduction of heat in *finite regions.*

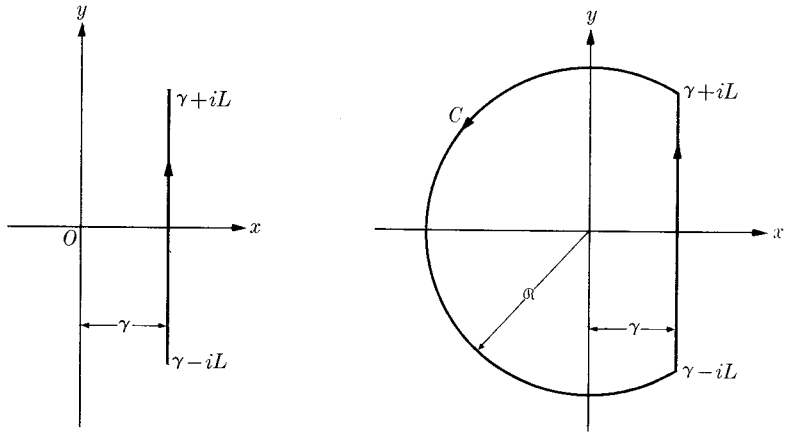
Let us illustrate the use of Eq. (7–229) by the inversion of

$$\frac{e^{x/p}}{p} = \sum_{n=0}^{\infty}\frac{x^n}{n!p^{n+1}}. \tag{7–230}$$

Since the $(n + 1)$th term of Eq. (7–230) has only a pole of order $(n + 1)$ at $z = 0$, it follows from Eq. (7–220) that

$$\text{Res}\,(0) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\left[\frac{1}{n!}\frac{d^n(e^{tz})}{dz^n}\right]_{z=0} = \sum_{n=0}^{\infty}\frac{(xt)^n}{(n!)^2}. \tag{7–231}$$

Furthermore, comparing Eq. (7–231) with the series expansion of the modified



(a) (b) Contour I

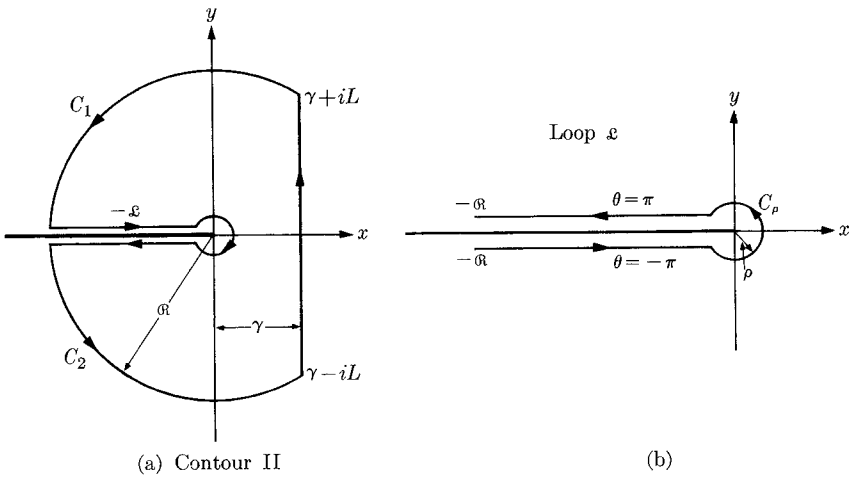
FIG. 7–20

Bessel function of the first kind, of order zero [see Eq. (7–84)], we have

$$e^{x/p}/p = L\{I_0[2(xt)^{1/2}]\}, \tag{7–79}$$

which is identical to TP(47) in Table 7–2. This transform pair plays an important role in the solution of unsteady problems related to single-fluid heat exchangers (see Examples 7–5 and 7–7).

(ii) *If $f(z)$ is analytic except for a number of poles,* all of which are to the left of some line $x = \gamma$, *and if it also has a branch point at* $z = 0$, then by introducing a cut along the negative real axis,† we complete the contour of the inversion integral by a large circle of radius $\mathfrak{R}$ which encloses all poles of the integrand, and a loop $\mathfrak{L}$ along the cut and around the branch point (Fig. 7–21).



(a) Contour II (b)

FIG. 7–21

Then Cauchy's residue theorem applied to the closed contour of Fig. 7–21(a), say *Contour II*, gives

$$\lim_{L\to\infty}\int_{\gamma-iL}^{\gamma+iL} e^{tz}\overline{T}(\mathbf{r}, z)\,dz + \lim_{\mathfrak{R}\to\infty}\int_{C_1+C_2} e^{tz}\overline{T}(\mathbf{r}, z)\,dz - \lim_{\substack{\mathfrak{R}\to\infty\\ \rho\to 0}}\int_{\mathfrak{L}} e^{tz}\overline{T}(\mathbf{r}, z)\,dz$$

$$= 2\pi i \lim_{N\to\infty}\sum_{n=1}^{N} \text{Res}\,(a_n). \tag{7–232}$$

Again, by use of Eq. (7–197), the integral over the circle $C_1 + C_2$ may be shown

† The selection of the negative real axis as the cut is arbitrary. As long as we stay on the left of $x = \gamma$, the direction of any radius or any curve connecting the origin to the circle may be used for this purpose.

to vanish as $\Re \to \infty$. Thus in the limit we get

$$T(\mathbf{r}, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma-i\infty} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = \sum_{n=1}^{\infty} \text{Res}\,(a_n) + \frac{1}{2\pi i}\lim_{\substack{\Re\to\infty \\ \rho\to 0}} \int_{\mathfrak{L}} e^{tz}\overline{T}(\mathbf{r}, z)\,dz. \tag{7–233}$$

For convenience, the loop integral of Eq. (7–233) may be expressed more explicitly; noting that $z = -\rho$ on both the lower and upper branches of the loop, we obtain

$$\int_{\mathfrak{L}} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = -\int_{\Re}^{\rho} e^{-t\rho}\overline{T}(\mathbf{r}, \rho e^{-i\pi})\,dz + \lim_{\rho\to 0}\int_{C_\rho} e^{tz}\overline{T}(\mathbf{r}, z)\,dz$$
$$- \int_{\rho}^{\Re} e^{-t\rho}\overline{T}(\mathbf{r}, \rho e^{i\pi})\,dz. \tag{7–234}$$

Since $\overline{T}(\mathbf{r}, z)$ is considered implicitly, its argument along the lower and upper branches of the loop must be kept as $z = \rho e^{-i\pi}$ and $z = \rho e^{i\pi}$, respectively. As $\Re \to \infty$ and $\rho \to 0$ Eq. (7–234) tends to

$$\lim_{\substack{\Re\to\infty \\ \rho\to 0}} \int_{\mathfrak{L}} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = \int_0^{\infty} e^{-t\rho}[\overline{T}(\mathbf{r}, \rho e^{-i\pi}) - \overline{T}(\mathbf{r}, \rho e^{i\pi})]\,d\rho$$
$$+ \lim_{\rho\to 0}\int_{C_\rho} e^{tz}\overline{T}(\mathbf{r}, z)\,dz. \tag{7–235}$$

In Eq. (7–235) the integral around the small circle C_ρ is left complex, to be evaluated individually for each particular problem. Note that ρ remains constant and the differential of $z = \rho e^{i\theta}$ is $dz = i\rho e^{i\theta}\,d\theta$ for this integral. Now, inserting Eq. (7–235) into Eq. (7–233), we get finally†

$$T(\mathbf{r}, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\overline{T}(\mathbf{r}, z)\,dz = \sum_{n=1}^{\infty} \text{Res}\,(a_n) + \frac{1}{2\pi i}\lim_{\rho\to 0}\int_{C_\rho} e^{tz}\overline{T}(\mathbf{r}, z)\,dz$$
$$+ \frac{1}{2\pi i}\int_0^{\infty} e^{-t\rho}[\overline{T}(\mathbf{r}, \rho e^{-i\pi}) - \overline{T}(\mathbf{r}, \rho e^{i\pi})]\,d\rho. \tag{7–236}$$

This case usually arises in problems involving conduction of heat in *semi-infinite regions*.

Let us illustrate the use of Eq. (7–236) by the inversion of

$$e^{-qx}/p, \tag{7–237}$$

where $q = (p/a)^{1/2}$. According to Definition (6) of Section 7–7, Eq. (7–237) is multiple-valued at $z = 0$. This may readily be seen from every other term

† If $\overline{T}(\mathbf{r}, z)$ does not have a branch point at $z = 0$, then Eq. (7–236) reduces to Eq. (7–229), as expected.

of its Maclaurin expansion when the argument of these terms is changed by 2π. Thus Eq. (7–237) has a branch point at $z = 0$. Since $z = 0$ must be excluded by a cut from the region in which the function is analytic, it cannot be a simple pole even though this makes the denominator of Eq. (7–237) zero. Thus the branch point $z = 0$ is the only singularity of Eq. (7–237). It follows from Eq. (7–236) that the inversion of Eq. (7–237) is

$$\frac{1}{2\pi i}\lim_{\rho\to 0}\int_{C\rho}[e^{tz-(z/a)^{1/2}x}]\frac{dz}{z} - \frac{1}{2\pi i}\int_0^\infty e^{-t\rho}[e^{-x(\rho/a)^{1/2}e^{-i\pi/2}} - e^{-x(\rho/a)^{1/2}e^{i\pi/2}}]\frac{d\rho}{\rho}. \tag{7–238}$$

Since the integrand of the first integral of Eq. (7–238) is continuous, taking the limit inside the integral sign,

$$\lim_{\rho\to 0}[e^{t\rho e^{i\theta}-x(\rho/a)^{1/2}e^{i\theta/2}}] \to 1,$$

noting that $dz/z = i\theta$, and inserting $e^{-i\pi/2} = -1$ and $e^{i\pi/2} = 1$ into the second integral, we get after rearranging

$$1 - \frac{1}{\pi}\int_0^\infty e^{-t\rho}\sin\left(\frac{\rho}{a}\right)^{1/2} x\left(\frac{d\rho}{\rho}\right) = \operatorname{erfc}\left[\frac{x}{2(at)^{1/2}}\right].$$

The equivalence of the result to the complementary error function may be seen by referring to a table of integrals. Thus we have shown by the inversion theorem that

$$\frac{e^{-qx}}{p} = L\left\{\operatorname{erfc}\left[\frac{x}{2(at)^{1/2}}\right]\right\}, \tag{7–239}$$

which is identical to TP(30) in Table 7–2. Equation (7–239) is an important transform pair in the solution of conduction problems in semi-infinite regions (see Examples 7–3 and 7–4).

If the singularities of $\overline{T}(\mathbf{r}, z)$ belong to neither one of the foregoing two cases, the inversion process may still be carried out by a similar procedure, provided another contour suitable to the new case is chosen. However, this often requires knowledge on the techniques of contour integration that is beyond the scope of this text.

The next section is devoted to problems whose solution by the inversion theorem leads to an integration along Contours I or II.

7–9. Solutions Obtainable by the Inversion Theorem

After reconsidering an example used earlier in this chapter, we shall employ some examples from Chapter 5, to illustrate the use of the inversion theorem.

Example 7–9. Let us find the solution of Example 7–2 by means of the inversion theorem.

The transforms of the lumped wall and lumped fluid temperatures have already been obtained as

$$\frac{\bar{\psi}(p)}{\psi_0} = \frac{p + \lambda_1 + \lambda_2}{(p + \lambda_1)(p + \lambda_2)}, \tag{7–37}$$

$$\frac{\bar{\phi}(p)}{\psi_0} = \frac{p + b_1 + b_2}{(p + \lambda_1)(p + \lambda_2)}. \tag{7–38}$$

We now employ the inversion theorem rather than the transform tables in the inversion of Eqs. (7–37) and (7–38). Thus, from Eq. (7–161), we get

$$\frac{\psi(t)}{\psi_0} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\frac{z + \lambda_1 + \lambda_2}{(z + \lambda_1)(z + \lambda_2)}\,dz \tag{7–240}$$

and

$$\frac{\phi(t)}{\psi_0} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\frac{z + b_1 + b_2}{(z + \lambda_1)(z + \lambda_2)}\,dz. \tag{7–241}$$

The only singularities of Eqs. (7–240) and (7–241) are the simple poles at $z = -\lambda_1$ and $z = -\lambda_2$. It follows from Contour I and Eq. (7–229) that

$$\begin{Bmatrix}\psi(t)/\psi_0\\ \phi(t)/\psi_0\end{Bmatrix} = \text{Res}\,(-\lambda_1) + \text{Res}\,(-\lambda_2). \tag{7–242}$$

Since the poles are simple, Eq. (7–242) may be rearranged by means of Eq. (7–221). The result is

$$\frac{\psi(t)}{\psi_0} = \left[e^{tz}\left(\frac{z + \lambda_1 + \lambda_2}{z + \lambda_2}\right)\right]_{z=-\lambda_1} + \left[e^{tz}\left(\frac{z + \lambda_1 + \lambda_2}{z + \lambda_1}\right)\right]_{z=-\lambda_2} \tag{7–243}$$

and

$$\frac{\phi(t)}{\psi_0} = \left[e^{tz}\left(\frac{z + b_1 + b_2}{z + \lambda_2}\right)\right]_{z=-\lambda_1} + \left[e^{tz}\left(\frac{z + b_1 + b_2}{z + \lambda_1}\right)\right]_{z=-\lambda_2}. \tag{7–244}$$

Thus from Eqs. (7–243) and (7–244) we directly obtain the solution of the problem in the expected form,

$$\frac{T(t) - T_\infty}{T_0 - T_\infty} = \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_1 - \lambda_2} \tag{7–41}$$

and

$$\frac{\theta(t) - T_\infty}{T_0 - T_\infty} = \frac{T(t) - T_\infty}{T_0 - T_\infty} - b_3\frac{e^{-\lambda_2 t} - e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}. \tag{7–42}$$

Example 7–10. Let us find the solution of Example 5–2 by the use of the inversion theorem.

The formulation of the problem, taken from Example 5–2, is

$$\frac{\partial\theta}{\partial t} = a\frac{\partial^2\theta}{\partial x^2}, \qquad \theta(x, 0) = \theta_0,$$

$$\frac{\partial\theta(0, t)}{\partial x} = 0, \qquad \theta(L, t) = 0.$$

The transformed differential equation and the transformed boundary conditions are readily found to be

$$\frac{d^2\bar{\theta}}{dx^2} - q^2\bar{\theta} = -\frac{\theta_0}{a}, \tag{7–245}$$

$$\frac{d\bar{\theta}(0, p)}{dx} = 0, \tag{7–246}$$

$$\bar{\theta}(L, p) = 0, \tag{7–247}$$

where, as before, $q = (p/a)^{1/2}$.

The solution of Eq. (7–245) that is satisfied by Eqs. (7–246) and (7–247) is

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{\cosh qx}{p\cosh qL}. \tag{7–248}$$

Since the second term in Eq. (7–248) does not appear in our tables of transforms, the use of the inversion theorem is necessary. Employing TP(1) for the first term and Eq. (7–162) for the second, we have

$$\frac{\theta(x, t)}{\theta_0} = 1 - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\frac{\cosh (z/a)^{1/2}x}{z\cosh (z/a)^{1/2}L}\,dz. \tag{7–249}$$

Although their arguments are double-valued at $z = 0$, $\cosh (z/a)^{1/2}x$ and $\cosh (z/a)^{1/2}L$, both being even functions, remain single-valued; they possess no branch points at $z = 0$ or elsewhere. Since z appears as a factor in the denominator of the integrand of Eq. (7–249), and $\cosh (z/a)^{1/2}L \to 1$ as $z \to 0$, $z = 0$ is a simple zero of the denominator and a simple pole of the integrand of Eq. (7–249). Furthermore, $\cosh (z/a)^{1/2}L$ appears, at first glance, to have no zeros. This is true, of course, when its argument is real. However, in the inversion theorem, z was not restricted to real values. It may be shown that $\cosh (z/a)^{1/2}L$ will continue to have no zeros for any complex argument, say $(z/a)^{1/2} = \mu + i\lambda$, but when the argument is pure imaginary, say $(z/a)^{1/2} = i\lambda$ or $z = -a\lambda^2$, noting from Eq. (7–168) that $\cosh i\lambda L = \cos \lambda L$, we find an infinite number of simple zeros for $\cosh (z/a)^{1/2}L$ corresponding to the values

$$\lambda_n L = (2n + 1)\pi/2, \qquad n = 0, 1, 2, \ldots \tag{7–250}$$

Thus the integrand of Eq. (7–249) has an infinite number of simple poles corresponding to the locations

$$z = -a[(2n + 1)\pi/2L]^2, \qquad n = 0, 1, 2, \ldots \tag{7–251}$$

along the negative real axis. Note that these poles are identical to the *characteristic values* of the same problem obtained by the use of separation of variables (see Example 5–2). This general fact will also be observed in the subsequent examples.

Since the integrand of Eq. (7–249) is analytic except for a number of poles, we employ Contour I and Eq. (7–229), and obtain

$$\frac{\theta(x, t)}{\theta_0} = 1 - \text{Res } (0) - \sum_{n=0}^{\infty} \text{Res } (-a\lambda_n^2). \tag{7–252}$$

It follows readily from Eq. (7–221) that the residue at the pole $z = 0$ is unity. The residue at the pole $z = -a\lambda_n^2$ may be obtained by comparing the integrand of Eq. (7–249) to Eq. (7–222); thus $P(z) = e^{tz} \cosh (z/a)^{1/2}x$, $k = 1$, and $Q(z) = \cosh (z/a)^{1/2}L$. Equation (7–252) may then be rearranged by using Eq. (7–225) for $k = 1$ to give

$$\frac{\theta(x, t)}{\theta_0} = -\sum_{n=0}^{\infty} \frac{P(-a\lambda_n^2)}{-a\lambda_n^2 (dQ/dz)_{z=-a\lambda_n^2}}. \tag{7–253}$$

Introducing the variable $\zeta = (z/a)^{1/2}L$ for convenience in the evaluation of dQ/dz, we may write Eq. (7–253) in the form

$$\frac{\theta(x, t)}{\theta_0} = \sum_{n=0}^{\infty} \frac{e^{-a\lambda_n^2 t} \cosh i\lambda_n x}{a\lambda_n^2 \left[\dfrac{d}{d\zeta} (\cosh \zeta) \dfrac{d\zeta}{dz}\right]_{z=-a\lambda_n^2}}. \tag{7–254}$$

The term in brackets in the denominator yields

$$\left\{\left[\sinh \left(\frac{z}{a}\right)^{1/2} L\right] \frac{L}{2(az)^{1/2}}\right\}_{z=-a\lambda_n^2} = (\sinh i\lambda_n L) \frac{L}{2ia\lambda_n}. \tag{7–255}$$

Now, inserting Eq. (7–255) into Eq. (7–254), and noting from Eq. (7–168) that $\cosh i\lambda_n x = \cos \lambda_n x$ and $\sinh i\lambda_n L = i \sin \lambda_n L$, and considering Eq. (7–250) to get $\sin \lambda_n L = (-1)^n$, we have finally

$$\frac{T(x, t) - T_\infty}{T_0 - T_\infty} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n L)} e^{-a\lambda_n^2 t} \cos \lambda_n x, \tag{5–10}$$

which is identical to the result of Example 5–2.

Example 7–11. Let us find the solution of Example 5–3 by means of the inversion theorem.

The only difference between the formulations of Examples 5–2 and 5–3 is in their surface boundary condition at $x = L$; the former formulation is based on a large heat transfer coefficient, the latter on a moderate one. Thus the transformed formulation from Example 7–10, modified to include the effect of a moderate h, readily gives

$$\frac{d^2\bar{\theta}}{dx^2} - q^2\bar{\theta} = -\frac{\theta_0}{a}, \tag{7–245}$$

$$\frac{d\bar{\theta}(0, p)}{dx} = 0, \tag{7–246}$$

$$-k\frac{d\bar{\theta}(L, p)}{dx} = h\bar{\theta}(L, p). \tag{7–256}$$

The solution of Eq. (7–245) that is satisfied by Eqs. (7–246) and (7–256) is

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{\cosh qx}{p[\cosh qL + (k/h)q \sinh qL]}. \tag{7–257}$$

Again, the use of TP(1) for the first term and Eq. (7–162) for the second gives

$$\frac{\theta(x, t)}{\theta_0} = 1 - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{tz}\cosh (z/a)^{1/2}x}{z[\cosh (z/a)^{1/2}L + (k/h)(z/a)^{1/2}\sinh (z/a)^{1/2}L]}\, dz. \tag{7–258}$$

Let us now follow the argument analogous to that of Example 7–10. Although $(z/a)^{1/2}$ is double-valued at $z = 0$, the functions $\cosh (z/a)^{1/2}x$, $\cosh (z/a)^{1/2}L$, and $(z/a)^{1/2}\sinh (z/a)^{1/2}L$ are single-valued, since they are all even functions; they possess no branch points at $z = 0$ or elsewhere. Since z appears as a factor in the denominator of the integrand of Eq. (7–258) and since for $z = 0$ the bracketed part of the denominator is not zero, $z = 0$ is a simple zero of the denominator and a simple pole of the integrand. It may also be shown that the bracketed part of the denominator does not have zeros for any real or complex argument except for the pure imaginary argument, say $(z/a)^{1/2} = i\lambda$ or $z = -a\lambda^2$, for which it may be rearranged as

$$(\lambda_n L)\sin \lambda_n L = \mathrm{Bi}\cos \lambda_n L. \tag{7–259}$$

Equation (7–259) has an infinite number of simple zeros along the negative real axis corresponding to locations $z = -a\lambda_n^2$. These zeros are the simple poles of the integrand, and are also identical to the characteristic values obtained in Example 5–3.

Like the integrand in Example 7–10, the integrand of Eq. (7–258) is analytic except for a number of simple poles, and it has the particular form of Eq. (7–222)

written for $k = 1$. Thus Eq. (7–253) is also valid for the present case. Introducing again the variable $\zeta = (z/a)^{1/2}L$, we may interpret Eq. (7–253) in terms of the integrand of Eq. (7–258) to give

$$\frac{\theta(x, t)}{\theta_0} = \sum_{n=1}^{\infty} \frac{e^{-a\lambda_n^2 t} \cosh i\lambda_n x}{a\lambda_n^2 \left[\dfrac{d}{d\zeta}\left(\cosh \zeta + \dfrac{\zeta}{\mathrm{Bi}} \sinh \zeta\right)\dfrac{d\zeta}{dz}\right]_{z=-a\lambda_n^2}}. \tag{7–260}$$

The terms in brackets in the denominator of Eq. (7–260) readily yield

$$\left(\sinh i\lambda_n L + \frac{1}{\mathrm{Bi}} \sinh i\lambda_n L + \frac{i\lambda_n L}{\mathrm{Bi}} \cosh i\lambda_n L\right) \frac{L}{2ia\lambda_n}. \tag{7–261}$$

Introducing Eq. (7–261) into Eq. (7–260), converting hyperbolic functions to circular functions, and eliminating the Biot modulus by means of Eq. (7–259), we have finally

$$\frac{T(x, t) - T_\infty}{T_0 - T_\infty} = 2 \sum_{n=1}^{\infty} \left(\frac{\sin \lambda_n L}{\lambda_n L + \sin \lambda_n L \cos \lambda_n L}\right) e^{-a\lambda_n^2 t} \cos \lambda_n x, \tag{5–13}$$

which is identical to the result of Example 5–3.

Example 7–12. Let us find the solution of Example 5–4 by the inversion theorem.

The formulation of the problem, taken from Example 5–4, is

$$\frac{\partial \theta}{\partial t} = a \frac{\partial^2 \theta}{\partial x^2} + \frac{u'''}{\rho c}, \qquad \theta(x, 0) = 0,$$

$$\frac{\partial \theta(0, t)}{\partial x} = 0, \qquad \theta(L, t) = 0.$$

As we learned in Chapters 4 and 5, the use of the separation of variables in the solution of the foregoing formulation would require the homogeneity of the associated differential equation through an appropriate superposition. Here and in the following examples we shall observe that the problem of nonhomogeneity in relation to governing equations and/or boundary conditions need not be considered when we are seeking a solution by means of Laplace transforms.

The transformed formulation of the problem is

$$\frac{d^2\bar{\theta}}{dx^2} - q^2\bar{\theta} = -\left(\frac{u'''}{k}\right)\frac{1}{p}, \tag{7–262}$$

$$\frac{d\bar{\theta}(0, p)}{dx} = 0, \tag{7–263}$$

$$\bar{\theta}(L, p) = 0. \tag{7–264}$$

The solution of Eq. (7–262) that satisfies Eqs. (7–263) and (7–264) may readily be found in the form

$$\frac{\bar{\theta}(x, p)}{u'''/\rho c} = \frac{1}{p^2} - \frac{\cosh qx}{p^2 \cosh qL}. \tag{7–265}$$

Thus TP(2) for the first term and Eq. (7–162) for the second yield

$$\frac{\theta(x, t)}{u'''/\rho c} = t - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\frac{\cosh (z/a)^{1/2}x}{z^2 \cosh (z/a)^{1/2}L}\,dz. \tag{7–266}$$

The integrand of Eq. (7–266) has a double pole at $z = 0$, and simple poles at $z = -a\lambda_n^2$, where $\lambda_n L = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ Since the integrand is analytic except for a number of poles, the use of Contour I again results in

$$\frac{\theta(x, t)}{u'''/\rho c} = t - \text{Res } (0) - \sum_{n=0}^{\infty} \text{Res } (-a\lambda_n^2). \tag{7–267}$$

The residue at $z = 0$ may be obtained from the series expansion of the integrand.† Thus, by the Maclaurin expansion of the numerator and the denominator, we have

$$\frac{e^{tz}\cosh (z/a)^{1/2}x}{z^2 \cosh (z/a)^{1/2}L} = \frac{(1 + tz + \cdots)(1 + zx^2/2a + \cdots)}{z^2(1 + zL^2/2a + \cdots)}.$$

Dividing now the expansion of $\cosh (z/a)^{1/2}x$ by the expansion of $\cosh (z/a)^{1/2}L$, and multiplying the result by $1/z^2$ and the expansion of e^{tz}, we get

$$\frac{1}{z^2} + \left[t - \frac{1}{2a}(L^2 - x^2)\right]\frac{1}{z} + \cdots$$

Then, according to Eqs. (7–215) and (7–218), the residue at the pole $z = 0$ is the coefficient of $1/z$, namely,

$$\text{Res } (0) = t - \frac{1}{2a}(L^2 - x^2). \tag{7–268}$$

Noting that the integrand of Eq. (7–266) now has the particular form of Eq. (7–222) for $k = 2$, we obtain the residue at the poles $z = -a\lambda_n^2$ from Eq. (7–225) as

$$\text{Res } (-a\lambda_n^2) = \frac{P(-a\lambda_n^2)}{(-a\lambda_n^2)^2 (dQ/dz)_{z=-a\lambda_n^2}}. \tag{7–269}$$

Further noting that $(dQ/dz)_{z=-a\lambda_n^2}$ of Eq. (7–269) is identical to that of Ex-

† The use of Eq. (7–220) for $k = 2$ would give the same result in a less convenient way.

ample 7–10 evaluated in Eq. (7–255), converting the hyperbolic functions to circular functions, and considering $\sin \lambda_n L = (-1)^n$, we find that

$$\text{Res}\,(-a\lambda_n^2) = \frac{2(-1)^n}{a\lambda_n^3 L} e^{-a\lambda_n^2 t} \cos \lambda_n x. \tag{7–270}$$

Introducing Eqs. (7–268) and (7–270) into Eq. (7–267), and rearranging the result, we have finally

$$\frac{\theta(x,t)}{u'''L^2/k} = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2\right] - 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n L)^3} e^{-a\lambda_n^2 t} \cos \lambda_n x, \tag{5–20}$$

which is identical to the result of Example 5–4.

Example 7–13. Let us find the solution of Example 5–5 by means of the inversion theorem.

Let us first recall that Example 5–5 illustrated unsteady problems whose solution does not approach a steady value for large values of time. To take care of nonhomogeneities of this type, we were forced to employ a superposition given by Eq. (5–21) rather than Eq. (5–14). We therefore concluded that the physics of a problem should be kept in mind when we seek a solution by means of the separation of variables. Utilizing the same example, we once more demonstrate that nonhomogeneities will not be an issue when the solution is sought by means of Laplace transforms.

From Example 5–5 we have the formulation of the problem as follows:

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}, \qquad \theta(x, 0) = 0,$$

$$\frac{\partial \theta(0,t)}{\partial x} = 0, \qquad +k\frac{\partial \theta(L,t)}{\partial x} = q''.$$

The transformed formulation is readily found to be

$$\frac{d^2\bar{\theta}}{dx^2} - q^2\bar{\theta} = 0, \tag{7–271}$$

$$\frac{d\bar{\theta}(0,p)}{dx} = 0, \tag{7–272}$$

$$\frac{d\bar{\theta}(L,p)}{dx} = \left(\frac{q''}{k}\right)\frac{1}{p}. \tag{7–273}$$

The solution of Eq. (7–271) that is satisfied by Eqs. (7–272) and (7–273) has the form

$$\frac{\bar{\theta}(x,p)}{q''/k} = \frac{\cosh qx}{qp \sinh qL}. \tag{7–274}$$

Then, according to the inversion theorem, we have

$$\frac{\theta(x,t)}{q''/k} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{tz}\frac{\cosh (z/a)^{1/2}x}{z(z/a)^{1/2}\sinh (z/a)^{1/2}L}\,dz. \tag{7–275}$$

Since $(z/a)^{1/2}L$ can be factored from the Maclaurin expansion of $\sinh (z/a)^{1/2}L$, $z(z/a)^{1/2}\sinh (z/a)^{1/2}L$ has a double zero and the integrand of Eq. (7–275) has a double pole at $z = 0$. Also, the integrand has simple poles, corresponding to the zeros of $\sinh (z/a)^{1/2}L$, at $z = -a\lambda_n^2$, where $\lambda_n L = n\pi$, $n = 1, 2, 3, \ldots$ Thus Contour I gives

$$\frac{\theta(x,t)}{q''/k} = \text{Res } (0) + \sum_{n=1}^{\infty} \text{Res } (-a\lambda_n^2). \tag{7–276}$$

In a manner similar to the evaluation of Res (0) in the previous example, we obtain from the expansion of the integrand

$$\frac{a/L}{z^2} + \left[\frac{at}{L} + \frac{1}{L}\left(\frac{x^2}{2} - \frac{L^2}{6}\right)\right]\frac{1}{z} + \cdots$$

It follows from the coefficient of $1/z$ that

$$\text{Res } (0) = \frac{at}{L} + \frac{1}{L}\left(\frac{x^2}{2} - \frac{L^2}{6}\right). \tag{7–277}$$

Comparing the integrand of Eq. (7–275) to Eq. (7–226) we note that $P(z) = e^{tz}\cosh (z/a)^{1/2}x$, $Q(z) = z(z/a)^{1/2}$, and $R(z) = \sinh (z/a)^{1/2}L$, and from Eq. (7–227) we obtain, corresponding to the zeros of $R(z)$,

$$\text{Res } (-a\lambda_n^2) = \frac{P(-a\lambda_n^2)}{Q(-a\lambda_n^2)(dR/dz)_{z=-a\lambda_n^2}}. \tag{7–278}$$

Furthermore, considering $(dR/dz)_{z=-a\lambda_n^2} = \cosh (z/a)^{1/2}L(L/2ia\lambda_n)$, expressing hyperbolic functions in terms of circular functions, and remembering that $\cos \lambda_n L = (-1)^n$, we may rearrange Eq. (7–278) to give

$$\text{Res } (-a\lambda_n^2) = -2L\frac{(-1)^n}{(\lambda_n L)^2}e^{-a\lambda_n^2 t}\cos \lambda_n x. \tag{7–279}$$

Introducing Eqs. (7–277) and (7–279) into Eq. (7–276), we have finally

$$\frac{\theta(x,t)}{q''L/k} = \frac{at}{L^2} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(\lambda_n L)^2}e^{-a\lambda_n^2 t}\cos \lambda_n x, \tag{5–33}$$

which is identical to the result of Example 5–5.

The solution of the differential formulation of Example 2–3 by Laplace transforms requires no mathematical clarification beyond that given in pre-

vious examples. We summarize by giving the solution of the problem in terms of the inversion theorem:

$$\frac{\theta(x,t)}{q''/k} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{tz}}{z}\left[\frac{(k/h)\cosh(z/a)^{1/2}L + (a/z)^{1/2}\sinh(z/a)^{1/2}L}{\cosh(z/a)^{1/2}L + (k/h)(z/a)^{1/2}\sinh(z/a)^{1/2}L} \times \cosh\left(\frac{z}{a}\right)^{1/2}x - \left(\frac{a}{z}\right)^{1/2}\sinh\left(\frac{z}{a}\right)^{1/2}x\right]dz. \quad (7\text{–}280)$$

The integrand of Eq. (7–280) has simple poles at $z = 0$ and at the zeros of the transcendental equation $(\lambda_n L)\sin\lambda_n L = \text{Bi}\cos\lambda_n L$. Then, from Contour I, we have

$$\frac{\theta(x,t)}{q''L/k} = 1 - \left(\frac{x}{L}\right) + \frac{1}{\text{Bi}} - 2\sum_{n=1}^{\infty}\frac{e^{-a\lambda_n^2 t}\cos\lambda_n x}{\lambda_n L(\lambda_n L + \sin\lambda_n L\cos\lambda_n L)}, \quad (5\text{–}34)$$

where x is measured from the bottom of the plate upward. The details of the solution above are left to the reader.

Example 7–14. Let us find the solution of Example 5–16 by the use of the inversion theorem.

As we learned in Chapter 5, the separation of variables cannot be directly employed for problems having time-dependent nonhomogeneities. In Example 5–16, we proceeded by first applying the separation of variables to the related problem resulting from a step function. Introducing the solution thus obtained into Duhamel's integral we obtained the solution of the original problem. The Laplace transforms in time, being directly applicable to time-dependent nonhomogeneities, circumvent this lengthy procedure.

From Example 5–16 we have the formulation of the problem in the form

$$\frac{\partial^2\theta_1}{\partial x^2} - m_1^2\theta_1 = \frac{1}{a}\frac{\partial\theta_1}{\partial t}, \qquad m_1^2 = 2h_1/k\delta,$$

$$\theta_1(x,0) = 0, \quad (5\text{–}119)$$

$$\frac{\partial\theta_1(0,t)}{\partial x} = 0, \qquad \theta_1(L,t) = \frac{q''}{h_2}(1 - e^{-bt}).$$

Then the transformed formulation is

$$\frac{d^2\bar{\theta}_1}{dx^2} - (m_1^2 + q^2)\bar{\theta}_1 = 0, \quad (7\text{–}281)$$

$$\frac{d\bar{\theta}_1(0,p)}{dx} = 0, \quad (7\text{–}282)$$

$$\bar{\theta}_1(L,p) = \frac{q''/\rho c\delta}{p(p+b)}. \quad (7\text{–}283)$$

The solution of Eq. (7–281) that satisfies Eqs. (7–282) and (7–283) is

$$\frac{\bar{\theta}_1(x, p)}{q''/\rho c\delta} = \frac{\cosh (m_1^2 + q^2)^{1/2}x}{p(p + b) \cosh (m_1^2 + q^2)^{1/2}L}. \tag{7–284}$$

Then, according to the inversion theorem,

$$\frac{\theta_1(x, t)}{q''/\rho c\delta} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{tz} \cosh (m_1^2 + z/a)^{1/2}x}{z(z + b) \cosh (m_1^2 + z/a)^{1/2}L} dz. \tag{7–285}$$

The integrand of Eq. (7–285) is analytic except for the simple poles at $z = 0$, $z = -b$, and $(m_1^2 + z/a)^{1/2} = i\lambda_n$ or $z = -a(m_1^2 + \lambda_n^2)$, where $\lambda_n L = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ It follows readily from Contour I that

$$\frac{\theta_1(x, t)}{q''/\rho c\delta} = \text{Res } (0) + \text{Res } (-b) + \sum_{n=0}^{\infty} \text{Res } [-a(m_1^2 + \lambda_n^2)]. \tag{7–286}$$

The residues of Eq. (7–286) may be evaluated in a manner similar to the previous examples. The results are†

$$\text{Res } (0) = \frac{\cosh m_1 x}{b \cosh m_1 L}, \tag{7–287}$$

$$\text{Res } (-b) = -\frac{e^{-bt} \cosh (m_1^2 - b/a)^{1/2}x}{b \cosh (m_1^2 - b/a)^{1/2}L}, \tag{7–288}$$

$$\text{Res } [-a(m_1^2 + \lambda_n^2)] = -\frac{2(-1)^n(\lambda_n L)e^{-a(m_1^2 + \lambda_n^2)t}}{a(m_1^2 + \lambda_n^2)L^2(m_1^2 + \lambda_n^2 - b/a)} \cos \lambda_n x. \tag{7–289}$$

Using Eqs. (7–287), (7–288), and (7–289) in Eq. (7–286) we have finally

$$\frac{\theta_1(x, t)}{q''/h_2} = \frac{\cosh m_1 x}{\cosh m_1 L} - \frac{e^{-bt} \cosh (m_1^2 - b/a)^{1/2}x}{\cosh (m_1^2 - b/a)^{1/2}L}$$

$$+ 2\left(\frac{b}{a}\right) \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda_n L)e^{-(m_1^2 + \lambda_n^2)t}}{(m_1^2 + \lambda_n^2)L^2(m_1^2 + \lambda_n^2 - b/a)} \cos \lambda_n x \tag{5–134}$$

which is identical to the result obtained in Example 5–16 by the separation of variables and Duhamel's superposition integral.

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† To obtain Eq. (7–289) assume that the integrand of Eq. (7–285) has the form of Eq. (7–226) so that $P(z) = e^{tz} \cosh (m_1^2 + z/a)^{1/2}x$, $Q(z) = z(z + b)$, and $R(z) = \cosh (m_1^2 + z/a)^{1/2}L$, then employ the second term of Eq. (7–227).

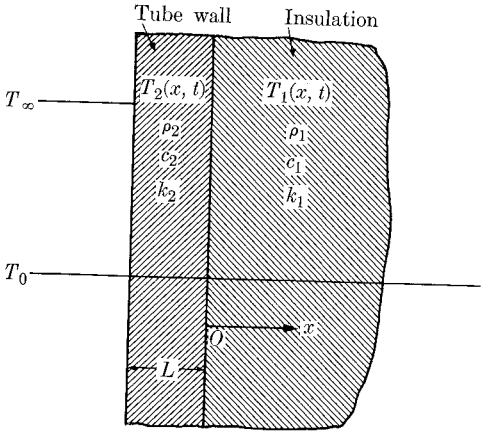


FIG. 7–22

Example 7–15. Neglecting the effect of curvature, we may treat a thick-walled insulated tube as a flat plate of thickness L attached to a semi-infinite solid (Fig. 7–22). (Here the insulation is assumed to have a large but finite thermal resistivity. Since the problem deals with the penetration of heat into the insulated tube, the effect of curvature may be neglected only in early stages of this penetration. Similarly, the assumption of infinite thickness of the insulation is valid only at early stages.) The initial temperature of the system is uniform, say T_0; then a condensing vapor at temperature T_∞ fills the tube instantaneously. The contact resistance between the plate and the solid is negligible. We wish to find the unsteady temperature of the system.

Measuring x most conveniently from the interface into the solid, we may write the formulation of the problem as follows:

$$\frac{\partial T_1}{\partial t} = a_1 \frac{\partial^2 T_1}{\partial x^2}, \qquad \frac{\partial T_2}{\partial t} = a_2 \frac{\partial^2 T_2}{\partial x^2},$$

$$T_1(x, 0) = T_2(x, 0) = T_0,$$

$$T_1(0, t) = T_2(0, t), \qquad k_1 \frac{\partial T_1(0, t)}{\partial x} = k_2 \frac{\partial T_2(0, t)}{\partial x}, \tag{7–290}$$

$$\lim_{x \to \infty} T_1(x, t) \to T_0, \qquad T_2(-L, t) = T_\infty.$$

Then in terms of $\theta_1 = T_1 - T_0$, $\theta_2 = T_2 - T_0$, $q_1^2 = p/a_1$, and $q_2^2 = p/a_2$, the transform of Eq. (7–290) becomes

$$\frac{d^2\bar{\theta}_1}{dx^2} - q_1^2\bar{\theta}_1 = 0, \qquad \frac{d^2\bar{\theta}_2}{dx^2} - q_2^2\bar{\theta}_2 = 0,$$

$$\bar{\theta}_1(0, p) = \bar{\theta}_2(0, p), \qquad k_1 \frac{d\bar{\theta}_1(0, p)}{dx} = k_2 \frac{d\bar{\theta}_2(0, p)}{dx}, \tag{7–291}$$

$$\lim_{x \to \infty} \bar{\theta}_1(x, p) \to 0, \qquad \bar{\theta}_2(-L, p) = \theta_\infty/p,$$

where $\theta_\infty = T_\infty - T_0$. The solution of Eq. (7–291) is

$$\frac{\bar{\theta}_1(x, p)}{\theta_\infty} = \frac{e^{-q_1 x}}{p(\cosh q_2 L + \epsilon \sinh q_2 L)}, \tag{7–292}$$

$$\frac{\bar{\theta}_2(x, p)}{\theta_\infty} = \frac{\cosh q_2 x - \epsilon \sinh q_2 x}{p(\cosh q_2 L + \epsilon \sinh q_2 L)}, \tag{7–293}$$

where $\epsilon = (\rho_1 c_1 k_1 / \rho_2 c_2 k_2)^{1/2}$. Taking Eq. (7–292) first, for example, we have from the inversion theorem

$$\frac{\theta_1(x, t)}{\theta_\infty} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{tz - (z/a_1)^{1/2} x}}{z[\cosh (z/a_2)^{1/2} L + \epsilon \sinh (z/a)^{1/2} L]} dz. \tag{7–294}$$

Since $e^{-(z/a_1)^{1/2} x}$ and $\sinh (z/a_2)^{1/2} L$ are double-valued, the integrand of Eq. (7–294) has a branch point at $z = 0$. (Show that there are no other singularities of the integrand.) Therefore, Contour II rather than Contour I is suitable to the evaluation of Eq. (7–294). Noting that the integrand possesses no poles, and that the integral around the small circle of loop $\mathcal{L}$ (Fig. 7–21b) gives $2\pi i$ in the limit as the integrand approaches $i\theta$ when $\rho \to 0$, we have from Eq. (7–236)

$$\begin{aligned} \frac{\theta_1(x, t)}{\theta_\infty} = 1 + \frac{1}{2\pi i} \int_0^\infty e^{-t\rho} \\ \times \left\{ \frac{e^{-x(\rho/a_1)^{1/2} e^{-i\pi/2}}}{\cosh [(\rho/a_2)^{1/2} e^{-i\pi/2} L] + \epsilon \sinh [(\rho/a_2)^{1/2} e^{-i\pi/2} L]} \right. \\ \left. - \frac{e^{-x(\rho/a_1)^{1/2} e^{i\pi/2}}}{\cosh [(\rho/a_2)^{1/2} e^{i\pi/2} L] + \epsilon \sinh [(\rho/a_2)^{1/2} e^{i\pi/2} L]} \right\} \frac{d\rho}{\rho}. \end{aligned} \tag{7–295}$$

Employing $\xi = (\rho/a_2)^{1/2}$, $\eta = (a_2/a_1)^{1/2}$, $e^{-i\pi/2} = -i$, and $e^{i\pi/2} = i$ in Eq. (7–295) and rearranging, we get

$$\frac{\theta_1(x, t)}{\theta_\infty} = 1 - \frac{2}{\pi} \int_0^\infty e^{-a_2 \xi^2 t} \left(\frac{\cos \xi L \sin \eta \xi x + \epsilon \sin \xi L \cos \eta \xi x}{\cos^2 \xi L + \epsilon^2 \sin^2 \xi L} \right) \frac{d\xi}{\xi}. \tag{7–296}$$

$\theta_2(x, t)$ may be obtained in a manner similar to $\theta_1(x, t)$. The details of the evaluation of $\theta_2(x, t)$ are left to the reader. The result is

$$\frac{\theta_2(x, t)}{\theta_\infty} = 1 - \epsilon \frac{2}{\pi} \int_0^\infty \frac{e^{-a_2 \xi^2 t} \sin \xi(x + L)}{\cos^2 \xi L + \epsilon^2 \sin^2 \xi L} \frac{d\xi}{\xi}. \tag{7–297}$$

Since the integrals of Eqs. (7–296) and (7–297) cannot be integrated explicitly, reference may be made to a numerical integration procedure for a parametric study of these integrals.

Let us now proceed to illustrative examples for cylindrical geometry.

Example 7–16. Let us find the solution of Example 5–7 by means of the inversion theorem.

The formulation of the problem, taken from Example 5–7, is

$$\frac{\partial\theta}{\partial t} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right), \qquad \theta(r,0) = \theta_0, \qquad \frac{\partial\theta(0,t)}{\partial r} = 0, \qquad \theta(R,t) = 0. \tag{5–45}$$

The transform of Eq. (5–45) is

$$\frac{d}{dr}\left(r\frac{d\bar{\theta}}{dr}\right) - q^2 r\bar{\theta} = \frac{\theta_0}{a}r, \tag{7–298}$$

$$\frac{d\bar{\theta}(0,p)}{dr} = 0, \tag{7–299}$$

$$\bar{\theta}(R,p) = 0. \tag{7–300}$$

The solution of the homogeneous part of Eq. (7–298) may be obtained from Table 3–1. Supplementing this solution by the particular solution, θ_0/p, and satisfying the result by Eqs. (7–299) and (7–300) gives

$$\frac{\bar{\theta}(r,p)}{\theta_0} = \frac{1}{p} - \frac{I_0(qr)}{pI_0(qR)}. \tag{7–301}$$

Introducing Eq. (7–301) into the inversion theorem we have

$$\frac{\theta(r,t)}{\theta_0} = 1 - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{tz}I_0[(z/a)^{1/2}r]}{zI_0[(z/a)^{1/2}R]}\,dz. \tag{7–302}$$

Although their arguments are double-valued, $I_0[(z/a)^{1/2}r]$ and $I_0[(z/a)^{1/2}R]$, both being even functions, are single-valued; they possess no branch points at $z = 0$. Moreover, for real and complex values of its argument $I_0[(z/a)^{1/2}R]$ possesses no zeros. However, for imaginary values of this argument, say $(z/a)^{1/2} = i\lambda$ or $z = -a\lambda^2$, $I_0[(z/a)^{1/2}R]$ is reduced to $J_0(\lambda R)$, which has an infinite number of simple zeros.† These zeros are identical to the characteristic values obtained in Example 5–7. Furthermore, we note that $I_0(0) = 1$, and conclude that $zI_0[(z/a)^{1/2}R]$ has a simple zero at $z = 0$ because of the factor z. Thus $z = 0$ and the zeros of $J_0(\lambda_n R) = 0$ are simple poles of the integrand of Eq. (7–302). It follows from Contour I that

$$\frac{\theta(r,t)}{\theta_0} = 1 - \text{Res}\,(0) - \sum_{n=1}^{\infty}\text{Res}\,(-a\lambda_n^2). \tag{7–303}$$

From Eq. (7–221) we get

$$\text{Res}\,(0) = 1. \tag{7–304}$$

† Note from Eq. (3–119) that $I_0(i\lambda R) = J_0(\lambda R)$.

Comparing the integrand of Eq. (7–302) to Eq. (7–222) for $k = 1$ yields $P(z) = e^{tz} I_0[(z/a)^{1/2} r]$ and $Q(z) = I_0[(z/a)^{1/2} R]$, and we obtain from Eq. (7–225)

$$\text{Res}\,(-a\lambda_n^2) = \frac{P(-a\lambda_n^2)}{-a\lambda_n^2 (dQ/dz)_{z=-a\lambda_n^2}}. \tag{7–305}$$

Next let us introduce the variable $\zeta = (z/a)^{1/2} R$; then, according to Eq. (3–137),

$$\left.\frac{dQ}{dz}\right|_{z=-a\lambda_n^2} = \frac{dI_0(\zeta)}{d\zeta} \left.\frac{d\zeta}{dz}\right|_{z=-a\lambda_n^2} = I_1(i\lambda_n R) \frac{R}{2ia\lambda_n}. \tag{7–306}$$

Now, inserting Eq. (7–306) into Eq. (7–305), we find the explicit form of the latter to be

$$\text{Res}\,(-a\lambda_n^2) = -\frac{2ie^{-a\lambda_n^2 t} I_0(i\lambda_n r)}{(\lambda_n R) I_1(i\lambda_n R)}. \tag{7–307}$$

Relating the modified Bessel functions to the Bessel functions by Eq. (3–119), and noting that the resulting Bessel functions $J_0(\lambda_n r)$ and $J_1(\lambda_n R)$ are even and odd functions, respectively, we may rearrange Eq. (7–307) to give

$$\text{Res}\,(-a\lambda_n^2) = -2 \frac{e^{-a\lambda_n^2 t} J_0(\lambda_n r)}{(\lambda_n R) J_1(\lambda_n R)}. \tag{7–308}$$

Finally, introducing Eqs. (7–304) and (7–308) into Eq. (7–303), we have

$$\frac{T(r, t) - T_\infty}{T_0 - T_\infty} = 2 \sum_{n=1}^{\infty} \frac{e^{-a\lambda_n^2 t} J_0(\lambda_n r)}{(\lambda_n R) J_1(\lambda_n R)}, \tag{5–51}$$

which is identical to the result of Example 5–7.

Example 7–17. Let us find the solution of Example 5–8 by means of Laplace transforms.

According to Example 5–8 the formulation of the problem is

$$\frac{\partial \theta}{\partial t} = \frac{a}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right), \quad \theta(r, 0) = \theta_0, \quad \frac{\partial \theta(0, t)}{\partial r} = 0, \quad -k \frac{\partial \theta(R, t)}{\partial r} = h\theta(R, t). \tag{5–52}$$

The transform of Eq. (5–52) gives

$$\frac{d}{dr}\left(r \frac{d\bar{\theta}}{dr}\right) - q^2 r \bar{\theta} = \frac{\theta_0}{a} r, \quad \frac{d\bar{\theta}(0, p)}{dr} = 0, \quad -k \frac{d\bar{\theta}(R, p)}{dr} = h\bar{\theta}(R, p). \tag{7–309}$$

The solution of Eq. (7–309) is

$$\frac{\bar{\theta}(r, p)}{\theta_0} = \frac{1}{p} - \frac{BI_0(qr)}{p[BI_0(qR) + qI_1(qR)]}, \tag{7–310}$$

where $B = h/k$. Employing the inversion theorem, we obtain then

$$\frac{\theta(r,t)}{\theta_0} = 1 - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{Be^{tz}I_0[(z/a)^{1/2}r]}{z\{BI_0[(z/a)^{1/2}R] + (z/a)^{1/2}I_1[(z/a)^{1/2}R]\}}\,dz. \tag{7–311}$$

The only singularities of the integrand of Eq. (7–311) are the simple poles at $z = 0$ and $z = -a\lambda_n^2$, where λ_n satisfies the transcendental equation

$$(\lambda_n R)J_1(\lambda_n R) - \text{Bi}\,J_0(\lambda_n R) = 0 \tag{7–312}$$

with $\text{Bi} = hR/k$. The use of Contour I results in Eq. (7–303) of the preceding example,

$$\frac{\theta(r,t)}{\theta_0} = 1 - \text{Res}\,(0) - \sum_{n=1}^{\infty} \text{Res}\,(-a\lambda_n^2). \tag{7–303}$$

Equation (7–305) of that example,

$$\text{Res}\,(-a\lambda_n^2) = \frac{P(-a\lambda_n^2)}{-a\lambda_n^2(dQ/dz)_{z=-a\lambda_n^2}}, \tag{7–305}$$

is also valid for the present case, provided $P(z) = Be^{tz}I_0[(z/a)^{1/2}r]$ and $Q(z) = BI_0[(z/a)^{1/2}R] + (z/a)^{1/2}I_1[(z/a)^{1/2}R]$. The residue at $z = 0$ is

$$\text{Res}\,(0) = 1. \tag{7–313}$$

For the residue at $z = -a\lambda_n^2$ we first calculate $(dQ/dz)_{z=-a\lambda_n^2}$. In terms of the variable $\zeta = (z/a)^{1/2}R$, we have

$$\left.\frac{dQ}{dz}\right|_{z=-a\lambda_n^2} = \left.\frac{dQ}{d\zeta}\,\frac{d\zeta}{dz}\right|_{z=-a\lambda_n^2} = [\text{Bi}\,I_1(i\lambda_n R) + i\lambda_n I_0(i\lambda_n R)]\,\frac{R}{2ia\lambda_n}. \tag{7–314}$$

Introducing Eq. (7–314) and the value of $P(z)$ into Eq. (7–305), rearranging the result by the transformation of the modified Bessel functions to the Bessel functions, and considering Eq. (7–312), we obtain

$$\text{Res}\,(-a\lambda_n^2) = \frac{2\text{Bi}\,e^{-a\lambda_n^2 t}J_0(\lambda_n r)}{(\lambda_n^2R^2 + \text{Bi}^2)J_0(\lambda_n R)}. \tag{7–315}$$

Finally, the use of Eqs. (7–313) and (7–315) in Eq. (7–303) gives

$$\frac{T(r,t) - T_\infty}{T_0 - T_\infty} = 2\,\text{Bi}\sum_{n=1}^{\infty} \frac{e^{-a\lambda_n^2 t}J_0(\lambda_n r)}{(\lambda_n^2R^2 + \text{Bi}^2)J_0(\lambda_n R)}, \tag{5–55}$$

which is identical to the result of Example 5–8.

Example 7–18. Consider a buried pipe (Fig. 7–23). The pipe and its surroundings have the same initial temperature, say T_∞. Then condensing steam at temperature T_0 suddenly fills the pipe. We wish to find the unsteady temperature of the system.

Noting that the heat transfer coefficient is large, and neglecting the temperature drop across the pipe wall because of its high thermal conductivity and small thickness, we find that the formulation of the problem in terms of $\theta = T - T_\infty$ is

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right), \qquad \theta(r, 0) = 0, \\ \theta(R, t) &= \theta_0, \qquad \lim_{r\to\infty} \theta(r, t) \to 0, \end{aligned} \tag{7–316}$$

where $\theta_0 = T_0 - T_\infty$. The transform of Eq. (7–316) is

$$\begin{aligned} \frac{d}{dr}\left(r\frac{d\bar{\theta}}{dr}\right) - q^2 r\bar{\theta} &= 0, \\ \bar{\theta}(R, p) &= \theta_0/p, \qquad \lim_{r\to\infty} \bar{\theta}(r, p) \to 0. \end{aligned} \tag{7–317}$$

The solution of Eq. (7–317) is

$$\frac{\bar{\theta}(r, p)}{\theta_0} = \frac{K_0(qr)}{pK_0(qR)}. \tag{7–318}$$

Then from the inversion theorem we have

$$\frac{\theta(r, t)}{\theta_0} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{tz}K_0[(z/a)^{1/2}r]}{zK_0[(z/a)^{1/2}R]}\,dz. \tag{7–319}$$

Since $(z/a)^{1/2}$ is double-valued, say $\rho^{1/2}e^{\pm i\pi/2}$, we obtain from the definition of $K_\nu(ze^{\pm i\pi/2})$,†

$$K_0(\rho^{1/2}e^{i\pi/2}r) = -i\frac{\pi}{2}[J_0(\rho^{1/2}r) - iY_0(\rho^{1/2}r)],$$

$$K_0(\rho^{1/2}e^{-i\pi/2}r) = i\frac{\pi}{2}[J_0(\rho^{1/2}r) + iY_0(\rho^{1/2}r)],$$

which implies that $K_0[(z/a)^{1/2}r]$ and $K_0[(z/a)^{1/2}R]$ also are double-valued. Hence $z = 0$ is a branch point of the integrand of Eq. (7–319). The integrand has no other singularities.

Note that the integral around the small circle (Contour C_ρ of Fig. 7–21b) gives $2\pi i$ in the limit as the integrand approaches $i\theta$ when $\rho \to 0$. Then the

† See Eq. (3–145).

use of Contour II and Eq. (7–236) results in

$$\frac{\theta(r,t)}{\theta_0} = 1 + \frac{1}{2\pi i}\int_0^\infty e^{-t\rho}\left\{\frac{K_0[(\rho/a)^{1/2}e^{-i\pi/2}r]}{K_0[(\rho/a)^{1/2}e^{-i\pi/2}R]} - \frac{K_0[(\rho/a)^{1/2}e^{i\pi/2}r]}{K_0[(\rho/a)^{1/2}e^{i\pi/2}R]}\right\}\frac{d\rho}{\rho}. \tag{7–320}$$

Introducing the variable $\xi = (\rho/a)^{1/2}$, and expressing K_0 in terms of J_0 and Y_0, we may rearrange Eq. (7–230) to give

$$\frac{\theta(r,t)}{\theta_0} = 1 + \frac{2}{\pi}\int_0^\infty e^{-a\xi^2 t}\left[\frac{J_0(\xi r)Y_0(\xi R) - J_0(\xi R)Y_0(\xi r)}{J_0^2(\xi R) + Y_0^2(\xi R)}\right]\frac{d\xi}{\xi}. \tag{7–321}$$

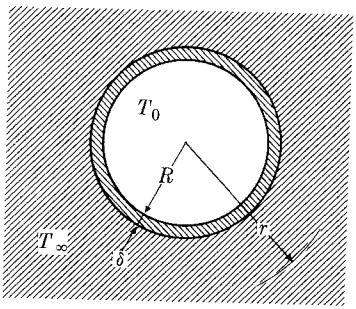


FIG. 7–23

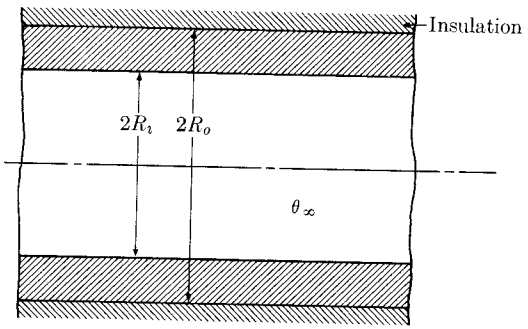


FIG. 7–24

Example 7–19. A condensing vapor at temperature θ_∞ suddenly fills an insulated pipe of inner and outer radii R_i and R_o, respectively (Fig. 7–24). The initial temperature of the pipe is zero. We wish to find the unsteady temperature of the pipe.

The formulation of the problem is

$$\frac{\partial\theta}{\partial t} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right), \qquad \theta(r,0) = 0,$$
$$\theta(R_i,t) = \theta_\infty, \qquad \frac{\partial\theta(R_o,t)}{\partial r} = 0. \tag{7–322}$$

The transform of Eq. (7–322) yields

$$\frac{d}{dr}\left(r\frac{d\bar{\theta}}{dr}\right) - q^2 r\bar{\theta} = 0,$$
$$\bar{\theta}(R_i,p) = \theta_\infty/p, \qquad \frac{d\bar{\theta}(R_o,p)}{dr} = 0. \tag{7–323}$$

The solution of Eq. (7–323) is found to be

$$\frac{\bar{\theta}(r,p)}{\theta_\infty} = \frac{K_1(qR_o)I_0(qr) + I_1(qR_o)K_0(qr)}{p[K_1(qR_o)I_0(qR_i) + I_1(qR_o)K_0(qR_i)]}. \tag{7–324}$$

From the inversion theorem we then have

$$\frac{\theta(r,t)}{\theta_\infty} = \frac{1}{2\pi i} \times \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{tz}\{K_1[(z/a)^{1/2}R_o]I_0[(z/a)^{1/2}r] + I_1[(z/a)^{1/2}R_o]K_0[(z/a)^{1/2}r]\}}{z\{K_1[(z/a)^{1/2}R_o]I_0[(z/a)^{1/2}R_i] + I_1[(z/a)^{1/2}R_o]K_0[(z/a)^{1/2}R_i]\}}\, dz. \tag{7–325}$$

Although $(z/a)^{1/2}$ is double-valued, say $\rho^{1/2}e^{\pm i\pi/2}$, it can be shown by relating the modified Bessel functions to the Bessel functions through the repeated use of Eqs. (3–144) and (3–145) that the integrand of Eq. (7–325) is single-valued. The only singularities of the integrand are the simple poles corresponding to $z = 0$ and to $z = -a\lambda_n^2$, where λ_n satisfies

$$Y_1(\lambda_n R_o)J_0(\lambda_n R_i) - J_1(\lambda_n R_o)Y_0(\lambda_n R_i) = 0. \tag{7–326}$$

Then Contour I gives

$$\frac{\theta(r,t)}{\theta_\infty} = \text{Res}\,(0) + \sum_{n=1}^{\infty} \text{Res}\,(-a\lambda_n^2). \tag{7–327}$$

The residue at $z = 0$ is†

$$\text{Res}\,(0) = 1. \tag{7–328}$$

The residue at $z = -a\lambda_n^2$ may be obtained from Eq. (7–305) of Example 7–15, namely,

$$\text{Res}\,(-a\lambda_n^2) = \frac{P(-a\lambda_n^2)}{-a\lambda_n^2(dQ/dz)_{z=-a\lambda_n^2}}, \tag{7–305}$$

provided P and Q are respectively assumed to be equal to the numerator and to the term in braces in the denominator of the integrand of Eq. (7–325). Here dQ/dz may be evaluated most conveniently in terms of the variable $q = (z/a)^{1/2}$. The result is

$$\frac{dQ}{dz} = \left(\frac{dQ}{dq}\right)\frac{dq}{dz} = \Big[-R_oK_0(qR_o)I_0(qR_i) - \frac{1}{q}K_1(qR_o)I_0(qR_i) + R_iK_1(qR_o)I_1(qR_i) + R_oI_0(qR_o)K_0(qR_i) - \frac{1}{q}I_1(qR_o)K_0(qR_i) - R_iI_1(qR_o)K_1(qR_i)\Big]\frac{1}{2a(z/a)^{1/2}}. \tag{7–329}$$

Now, relating the modified Bessel functions to the Bessel functions, inserting

† First note from Section 3–7 that for small arguments $I_0(z) \sim 1$, $I_1(z) \sim z/2$, $K_0(z) \sim -\ln z$, and $K_1(z) \sim 1/z$; then take the limit by letting $z \to 0$.

$i\lambda_n$ in place of $q = (z/a)^{1/2}$, then simplifying the result by means of Eq. (7–326), we have from Eq. (7–329)

$$\left(\frac{dQ}{dz}\right)_{z=-a\lambda_n^2} = \frac{\pi}{2}[R_o Y_0(\lambda_n R_o)J_0(\lambda_n R_i) - R_o J_0(\lambda_n R_o)Y_0(\lambda_n R_i) - R_i Y_1(\lambda_n R_o)J_1(\lambda_n R_i) + R_i J_1(\lambda_n R_o)Y_1(\lambda_n R_i)]\frac{1}{2ia\lambda_n}. \tag{7–330}$$

Similarly,

$$P(-a\lambda_n^2) = i\frac{\pi}{2}e^{-a\lambda_n^2 t}[Y_1(\lambda_n R_o)J_0(\lambda_n r) - J_1(\lambda_n R_o)Y_0(\lambda_n r)]. \tag{7–331}$$

Finally, the use of Eqs. (7–330) and (7–331) in Eq. (7–305), and the result together with Eq. (7–328) in Eq. (7–327) gives

$$\frac{\theta(r,t)}{\theta_\infty} = 1 - 2\sum_{n=1}^{\infty}\frac{e^{-a\lambda_n^2 t}[Y_1(\lambda_n R_o)J_0(\lambda_n r) - J_1(\lambda_n R_o)Y_0(\lambda_n r)]}{(\lambda_n R_o)\,\Delta_0(R_o, R_i) - (\lambda_n R_i)\,\Delta_1(R_o, R_i)}, \tag{7–332}$$

where

$$\Delta_0(R_o, R_i) = Y_0(\lambda_n R_o)J_0(\lambda_n R_i) - J_0(\lambda_n R_o)Y_0(\lambda_n R_i),$$
$$\Delta_1(R_o, R_i) = Y_1(\lambda_n R_o)J_1(\lambda_n R_i) - J_1(\lambda_n R_o)Y_1(\lambda_n R_i).$$

Since the well-known transformation $\theta(r, t) = \psi(r, t)/r$ [see Eq. (5–60)] reduces the problems of one-dimensional spherical geometry to those of cartesian geometry, examples illustrating spherical geometry are not considered here.†

In the next two examples we demonstrate the use of the inversion theorem in two-dimensional geometry.

Example 7–20. Let us find the solution of Example 5–11 by means of the inversion theorem.

The formulation of the problem, taken from Example 5–11, is

$$\frac{\partial\theta}{\partial t} = a\left(\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial y^2}\right), \qquad \theta(\xi, y, 0) = 0, \qquad \theta(0, y, t) = 0, \qquad \theta(L, y, t) = \theta_0,$$
$$\frac{\partial\theta(\xi, 0, t)}{\partial y} = 0, \qquad \theta(\xi, l, t) = 0. \tag{5–71}$$

The transformation of Eq. (5–71) gives

$$\frac{\partial^2\bar{\theta}}{\partial\xi^2} + \frac{\partial^2\bar{\theta}}{\partial y^2} - q^2\bar{\theta} = 0, \qquad \bar{\theta}(0, y, p) = 0, \qquad \bar{\theta}(L, y, p) = \frac{\theta_0}{p},$$
$$\frac{\partial\bar{\theta}(\xi, 0, p)}{\partial y} = 0, \qquad \bar{\theta}(\xi, l, p) = 0. \tag{7–333}$$

† See, however, Problems 7–28, 7–29, and 7–30.

Note that the use of Laplace transforms reduces the formulation of a multi-dimensional problem, a partial differential equation, to another partial differential equation involving one less independent variable, rather than to an ordinary differential equation as in one-dimensional problems. The solution of this transformed formulation requires further considerations. Equation (7–333), for example, may be solved by the separation of variables in a manner similar to steady problems of two-dimensional cartesian geometry.† The steps of the solution of Eq. (7–333) by the separation of variables, being trivial at this point, are not given. The result is

$$\frac{\bar{\theta}(\xi, y, p)}{\theta_0} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n \sinh (\lambda_n^2 + q^2)^{1/2}\xi}{(\lambda_n l) \sinh (\lambda_n^2 + q^2)^{1/2} L} \cos \lambda_n y, \tag{7–334}$$

where $\lambda_n l = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ Then from the inversion theorem we have

$$\frac{\theta(\xi, y, t)}{\theta_0} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \left[\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{tz} \sinh (\lambda_n^2 + z/a)^{1/2}\xi}{z \sinh (\lambda_n^2 + z/a)^{1/2} L} dz \right] \cos \lambda_n y. \tag{7–335}$$

The only singularities of the integrand of Eq. (7–335) are the simple poles corresponding to $z = 0$ and to

$$(\lambda_n^2 + z/a)^{1/2} = i\mu_m \qquad \text{or} \qquad z = -a(\lambda_n^2 + \mu_m^2),$$

where $\mu_m L = m\pi$, $m = 1, 2, 3, \ldots$ It follows then from Contour I that

$$\frac{\theta(\xi, y, t)}{\theta_0} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \left\{ \text{Res } (0) + \sum_{m=1}^{\infty} \text{Res } [-a(\lambda_n^2 + \mu_m^2)] \right\} \cos \lambda_n y. \tag{7–336}$$

The evaluation of residues is straightforward. The result is

$$\text{Res } (0) = \frac{\sinh \lambda_n \xi}{\sinh \lambda_n L}, \tag{7–337}$$

$$\text{Res } [-a(\lambda_n^2 + \mu_m^2)] = 2 \frac{(-1)^m (\mu_m L) e^{-a(\lambda_n^2 + \mu_m^2)t}}{(\lambda_n^2 + \mu_m^2) L^2} \sin \mu_m \xi. \tag{7–338}$$

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† This, of course, implies the existence of a more general method of solution, the successive use of two or more solution methods which are separately known to exist. A full discussion on this matter requires further background on the transform calculus and other solution techniques, and hence is beyond the scope of the text. See, however, Problem 7–14, involving the successive use of Laplace transforms.

The use of Eqs. (7–337) and (7–338) in Eq. (7–336) gives finally

$$\frac{\theta(\xi, y, t)}{\theta_0} = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \left(\frac{\sinh \lambda_n \xi}{\sinh \lambda_n L}\right) \cos \lambda_n y$$

$$+ 4\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (-1)^{n+m} \left(\frac{\mu_m L}{\lambda_n l}\right) \frac{e^{-a(\lambda_n^2+\mu_m^2)t}}{(\lambda_n^2 + \mu_m^2)L^2} \sin \mu_m \xi \cos \lambda_n y, \qquad (7\text{–}339)$$

which is identical to the result of Example 5–11, the sum of Eqs. (4–123) and (5–80).

We now have obtained the solution of Example 5–11 both by the separation of variables and by the successive use of Laplace transforms and separation of variables. The importance of the latter procedure will become evident in the next example, for which the former method is not suitable (why?).

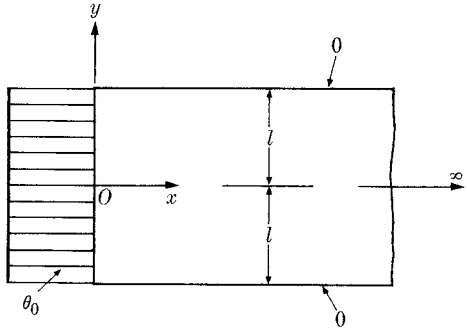


FIG. 7–25

Example 7–21. Let us reconsider the previous example, assuming now that the geometry of the problem extends to infinity in the x-direction.† We wish to find the unsteady temperature of the fin.

The formulation of the problem in terms of the coordinate system shown in Fig. 7–25 is

$$\frac{\partial \theta}{\partial t} = a\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right), \qquad \theta(x, y, 0) = 0, \qquad \theta(0, y, t) = \theta_0, \qquad \theta(\infty, y, t) = 0,$$

$$\frac{\partial \theta(x, 0, t)}{\partial y} = 0, \qquad \theta(x, l, t) = 0. \qquad (7\text{–}340)$$

The transform of Eq. (7–340) is readily found to be

$$\frac{\partial^2 \bar{\theta}}{\partial x^2} + \frac{\partial^2 \bar{\theta}}{\partial y^2} - q^2 \bar{\theta} = 0, \qquad \bar{\theta}(0, y, p) = \frac{\theta_0}{p}, \qquad \bar{\theta}(\infty, y, p) = 0,$$

$$\frac{\partial \bar{\theta}(x, 0, p)}{\partial y} = 0, \qquad \bar{\theta}(x, l, p) = 0. \qquad (7\text{–}341)$$

† As $t \to \infty$ the problem approaches Example 4–2.

Proceeding as in Example 7–20, and applying the separation of variables to Eq. (7–341), we get

$$\frac{\bar{\theta}(x, y, p)}{\theta_0} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \frac{e^{-(\lambda_n^2+q^2)^{1/2}x}}{p} \cos \lambda_n y, \tag{7–342}$$

where $\lambda_n l = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$ The inversion theorem is not needed for the inversion of Eq. (7–342). In fact, TP(28) appropriately combined with TP(IV) and TP(VI) gives

$$\frac{\theta(x, y, t)}{\theta_0} = \frac{2}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \left[\int_0^t \left(\frac{x^2}{4a\tau} \right)^{1/2} e^{-(a\lambda_n^2\tau + x^2/4a\tau)} \frac{d\tau}{\tau} \right] \cos \lambda_n y. \tag{7–343}$$

This example completes our study of the exact solution of unsteady problems by Laplace transforms. The next section is devoted to approximate solutions that are valid for either small or large values of time and that are obtained by considering the appropriate series expansions of the transformed solution. The methods to be developed are general, and may be applied to suitable space variables as well; cases of the latter however, occur less frequently than expansions in time.

7–10. Solutions Valid for Small or Large Values of Time

For small values of time the series solutions of unsteady transient problems,† obtained in Chapter 5 by the separation of variables and in Section 7–9 by the inversion theorem, are usually slowly convergent and not suitable for numerical computations. Thus, although these solutions can conveniently be used for large values of time, it becomes necessary to supplement them by other solutions which converge rapidly for small values of time. Furthermore, as illustrated by Example 5–6, the method of separation of variables is not suitable to unsteady periodic problems, and the method of complex temperature given in Chapter 6 is useful only for steady periodic solutions. In this section we employ Laplace transforms to find solutions valid for small or large values of time, and transient periodic solutions. Actually, transient periodic solutions could have been obtained in Section 7–9 by the inversion theorem; the reason for leaving these solutions to this section is that their steady part is best explained in terms of solutions valid for large values of time.

Thus the present section is devoted to a systematic study of solutions convenient for small or large values of time. Despite their convenience, the basic concept behind these solutions is surprisingly simple; it amounts to an appropriate series expansion of the transformed solution in terms of the Laplace

† Refer to Fig. 5–1 for the classification of unsteady problems.

transform parameter p. To explain the point further, let us first consider the definition of Laplace transforms,

$$L\{f(t)\} = \bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt, \tag{7–4}$$

and note the fact that p is, generally, a finite parameter which may be restricted, depending on the particular function $f(t)$ under consideration. Next, by interpreting Eq. (7–4) as an ordinary integral having a finite integrand, we conclude that the product pt must be finite. Therefore

$$\text{small } t \rightarrow \text{large } p,$$
$$\text{large } t \rightarrow \text{small } p.$$

Consequently, the expansion of the transformed solution of a problem into a convergent series of ascending powers of $1/p$ or p, and subsequent term-by-term inversion gives a solution useful for small or large values of time, respectively. Whenever convenient, the same concept may of course be applied to generalized expansions in terms of $1/f(p)$ or of $g(p)$, where $f(p)$ and $g(p)$ both are increasing functions of p, to be determined according to the particular problem in question. These expansions are demonstrated below by a number of examples.

1. *Solutions useful for small values of time.* Let us start with the lumped formulation of a problem. Assume that the transformed solution $\bar{T}(p)$ of such a formulation can be expanded into a power series in the form

$$\bar{T}(p) = \sum_{n=1}^{\infty} \frac{a_n}{p^n}. \tag{7–344}$$

The direct use of the inversion theorem or TP(3) readily gives the inversion of Eq. (7–344) as

$$T(t) = \sum_{n=1}^{\infty} \frac{a_n t^{n-1}}{(n-1)!}. \tag{7–345}$$

It is clear that Eq. (7–345) converges rapidly for small values of time. Since the transformed solution of lumped problems always has the form of a rational function of p, it can always be expanded into a convergent series like Eq. (7–344), leading to a solution in the form of Eq. (7–345).

Example 7–22. Let us find a solution of the lumped formulation of Example 2–2 useful for small values of time.

According to Example 2–2 the lumped formulation of the problem is

$$\rho c L \frac{dT}{dt} = -h(T - T_\infty) + u'''L, \tag{2–149}$$

$$T(0) = T_\infty. \tag{2–147}$$

Simplifying Eqs. (2–147) and (2–149) in terms of $\theta = T - T_\infty$, then taking their transforms, we obtain

$$\bar{\theta}(p) = \frac{n}{p(p + m)}, \tag{7–346}$$

where $m = h/\rho cL$ and $n = u'''/\rho c$. Equation (7–346), first rearranged, then expanded in powers of $1/p$, gives

$$\bar{\theta}(p) = \frac{n}{p^2(1 + m/p)} = n\left(\frac{1}{p^2} - \frac{m}{p^3} + \frac{m^2}{p^4} - \cdots\right). \tag{7–347}$$

Finally, term-by-term inversion of Eq. (7–347) yields a rapidly converging solution for small values of time in the form

$$\frac{T(t) - T_\infty}{u'''L/h} = mt - \frac{1}{2!}(mt)^2 + \frac{1}{3!}(mt)^3 - \cdots \tag{7–348}$$

Note that Eq. (7–348) is the Maclaurin expansion of the complete solution given by Eq. (2–150).

For distributed problems over a finite region, the transform solutions contain hyperbolic or cylindrical (Bessel) functions of $(p/a)^{1/2}$ or more complicated arguments. These solutions can more conveniently be expanded in series of negative exponentials rather than in powers of $1/p$. Then term-by-term inversion gives solutions valid for small values of time in terms of exponentials and error functions rather than in powers of t.

For cartesian geometry the foregoing method usually yields one or the other of two types of solutions, depending on the boundary conditions of the problem. (i) For simple boundary conditions, such as prescribed temperature or prescribed heat flux, all terms of the expansion have simple coefficients, leading to solutions which are useful for all values of time and not merely for small values of time. (ii) For more complicated boundary conditions, such as heat transfer to the ambient by convection, the coefficients of the expansion are progressively more and more complicated functions of negative exponentials, so that only the inversion of the first few terms can be obtained. The resulting solutions are therefore useful only for relatively small values of time. For other geometries even the simplest boundary conditions yield series of progressively more complicated terms. Solutions thus obtained are again valid for only relatively small values of time.

Example 7–23. Let us find a solution of Example 7–10 useful for small values of time.

The solution of the transformed formulation may be taken from Example 7–10. Thus

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{\cosh qx}{p \cosh qL}. \tag{7–248}$$

Expressing the hyperbolic functions in terms of exponentials, we obtain from Eq. (7–248)

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{e^{qx} + e^{-qx}}{p(e^{qL} + e^{-qL})}. \tag{7–349}$$

Further rearrangement of Eq. (7–349) by factoring the denominator of the second term on the right by e^{qL} gives

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{e^{qx} + e^{-qx}}{pe^{qL}(1 + e^{-2qL})}. \tag{7–350}$$

Noting that

$$\frac{1}{1 + e^{-2qL}} = \sum_{n=0}^{\infty} (-1)^n e^{-2qnL}, \tag{7–351}$$

we may write Eq. (7–350) in the form

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \sum_{n=0}^{\infty} \frac{(-1)^n}{p} \{e^{-[(2n+1)L-x]q} + e^{-[(2n+1)L+x]q}\}. \tag{7–352}$$

Then the successive use of TP(30) readily gives the inversion of Eq. (7–352) as

$$\frac{T(x, t) - T_\infty}{T_0 - T_\infty} = 1 - \sum_{n=0}^{\infty} (-1)^n \left\{\operatorname{erfc}\left[\frac{(2n + 1) - x/L}{2(at/L^2)^{1/2}}\right] - \operatorname{erfc}\left[\frac{(2n + 1) + x/L}{2(at/L^2)^{1/2}}\right]\right\}. \tag{7–353}$$

Equation (7–353) is convergent for all values of time. When the dimensionless number, the so-called Fourier modulus $\mathrm{Fo} = at/L^2$, is less than 1 only the first two or three terms of Eq. (7–353) are needed for four-place accuracy; the other solution of the problem, given by Eq. (5–10), would require the consideration of hundreds of terms for the same order of accuracy.† On the other hand, for large values of Fo Eq. (7–353) is slowly convergent, yet the first few terms of Eq. (5–10) provide a sufficiently accurate answer. Thus for numerical computations Eqs. (5–10) and (7–353) complement each other, one being suitable to small values of time and the other to large values of time.

Example 7–24. Let us find a solution of Example 7–11 valid for small values of time.

According to Example 7–11 the solution of the transformed formulation is

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{\cosh qx}{p[\cosh qL + (k/h)q \sinh qL]}. \tag{7–257}$$

† Equation (5–10) is also convergent for all values of time.

Equation (7–257) may be expressed in terms of the exponential functions to give

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{e^{qx} + e^{-qx}}{p[(e^{qL} + e^{-qL}) + (k/h)q(e^{qL} - e^{-qL})]}. \tag{7–354}$$

Now, factoring the bracketed portion by e^{qL} and rearranging, we obtain

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{e^{-(L-x)q} + e^{-(L+x)q}}{p[(1 + qk/h) + (1 - qk/h)e^{-2qL}]}. \tag{7–355}$$

Finally, factoring the bracketed terms of Eq. (7–355) by $(1 + qk/h)$, we get

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{e^{-(L-x)q} + e^{-(L+x)q}}{p(1 + qk/h)[1 + (1 - qk/h)e^{-2qL}/(1 + qk/h)]}, \tag{7–356}$$

and noting that

$$\frac{1}{1 + (1 - qk/h)e^{-2qL}/(1 + qk/h)} = 1 - \left(\frac{1 - qk/h}{1 + qk/h}\right)e^{-2qL} + \cdots,$$

we may further rearrange Eq. (7–356) to give

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{h/k}{p(h/k + q)}[e^{-(L-x)q} + e^{-(L+x)q}] + \frac{(h/k)(h/k - q)}{p(h/k + q)^2}[e^{-(3L-x)q} + e^{-(3L+x)q}] + \cdots \tag{7–357}$$

Employing TP(1) and TP(37) we get the inversion of the first two terms of Eq. (7–357) in the form

$$\frac{T(x, t) - T_\infty}{T_0 - T_\infty} = 1 - \left\{\operatorname{erfc}\left[\frac{1 - x/L}{2(at/L^2)^{1/2}}\right] + \operatorname{erfc}\left[\frac{1 + x/L}{2(at/L^2)^{1/2}}\right]\right\} + e^{(hL/k)(1-x/L)+(hL/k)^2(at/L^2)} \operatorname{erfc}\left[\left(\frac{hL}{k}\right)\left(\frac{at}{L^2}\right)^{1/2} + \frac{1 - x/L}{2(at/L^2)^{1/2}}\right] + e^{(hL/k)(1+x/L)+(hL/k)^2(at/L^2)} \operatorname{erfc}\left[\left(\frac{hL}{k}\right)\left(\frac{at}{L^2}\right)^{1/2} + \frac{1 + x/L}{2(at/L^2)^{1/2}}\right] + \cdots \tag{7–358}$$

Because of the boundary condition given by Eq. (7–256), the terms of Eq. (7–357) become progressively more complicated algebraically. Consequently, we have considered the inversion of only the first two terms of this equation, and hence Eq. (7–358) is valid for relatively small values of time only.

The next example, taken from cylindrical geometry, demonstrates that even the simplest boundary conditions employed with this geometry result in series having progressively more complex terms.

Example 7–25. Let us find a solution of Example 7–16 useful for small values of time.

The solution of the transformed formulation may be taken from Example 7–16. Thus

$$\frac{\bar{\theta}(r, p)}{\theta_0} = \frac{1}{p} - \frac{I_0(qr)}{pI_0(qR)}. \tag{7–301}$$

Remembering that small values of time correspond to large p (or q), we employ the asymptotic expansions of $I_0(qr)$ and $I_0(qR)$ [see Eq. (3–156)], and obtain

$$\frac{\bar{\theta}(r, p)}{\theta_0} = \frac{1}{p} - \frac{[e^{qr}/(2\pi qr)^{1/2}](1 + 1/8qr + 9/128q^2r^2 + \cdots)}{p[e^{qR}/(2\pi qR)^{1/2}](1 + 1/8qR + 9/128q^2R^2 + \cdots)}. \tag{7–359}$$

Dividing the numerator by the denominator, we may rearrange Eq. (7–359) in the form

$$\frac{\bar{\theta}(r, p)}{\theta_0} = \frac{e^{-q(R-r)}}{(r/R)^{1/2}p}\left(1 + \frac{R - r}{8Rrq} + \frac{9R^2 - 2Rr - 7r^2}{128R^2r^2q^2} + \cdots\right). \tag{7–360}$$

The use of TP(30), TP(31), and TP(32) in Eq. (7–360) gives

$$\begin{aligned}\frac{T(r, t) - T_\infty}{T_0 - T_\infty} = {} & \frac{1}{(r/R)^{1/2}} \operatorname{erfc}\left[\frac{1 - r/R}{2(at/R^2)^{1/2}}\right] \\ & + \frac{(1 - r/R)(at/R^2)^{1/2}}{4(r/R)^{3/2}} i \operatorname{erfc}\left[\frac{1 - r/R}{2(at/R^2)^{1/2}}\right] \\ & + \frac{9 - 2(r/R) - 7(r/R)^2}{32(r/R)^{5/2}}\left(\frac{at}{R^2}\right)^{1/2} i^2 \operatorname{erfc}\left[\frac{1 - r/R}{2(at/R^2)^{1/2}}\right] + \cdots,\end{aligned} \tag{7–361}$$

where $i \operatorname{erfc} x = \int_x^\infty \operatorname{erfc} \xi \, d\xi$ and $i^2 \operatorname{erfc} x = \int_x^\infty i \operatorname{erfc} \xi \, d\xi$. Since we have used the asymptotic expansion of the Bessel functions in Eq. (7–359), the validity of Eq. (7–361) for small values of time must be so restricted that r/R is not too small, and qr always remains large (see Problem 7–38).

Again, the problems of spherical geometry are not considered because they may be reduced to those of cartesian geometry by the transformation $\theta(r, t) = \psi(r, t)/r$.

Since the solutions obtained by the separation of variables or by the inversion theorem converge rapidly for large values of time, and those obtained by the expansion of the solution of the transformed formulation in powers of $1/p$ converge rapidly for small values of time, the simultaneous use of both types of solution would be convenient and sufficient for numerical investigation of our problems. It remains to be determined, however, what happens when we expand the solution of a transformed formulation in powers of p rather than in

powers of $1/p$. We know, of course, that the new solution would be valid for *small* $p \rightarrow$ *large* t. What we are interested in is the availability of this solution; this will be discussed next.

2. *Solutions useful for large values of time.* First, three illustrative examples are worked out; later the procedure is generalized.

Let us reconsider Example 7–22, and the solution of its transformed formulation

$$\bar{\theta}(p) = \frac{n}{p(p + m)}. \tag{7–346}$$

Equation (7–346) can be readily expanded in powers of p in the form

$$\bar{\theta}(p) = \frac{n}{mp(1 + p/m)} = \frac{n}{m}\left(\frac{1}{p} - \frac{1}{m} + \frac{p}{m^2} - \frac{p^2}{m^3} + \cdots\right). \tag{7–362}$$

Noting that all terms of the series of Eq. (7–362) except the first one are analytic, we get immediately from the inversion theorem

$$T(t) - T_\infty = n/m = u'''L/h,$$

which is the steady solution of the problem. Hence in place of an asymptotic solution, which would give the behavior of the present problem for large values of time, we obtained the asymptote, namely, the steady solution itself.

Second, let us reconsider Example 7–10, whose transformed solution is

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{\cosh qx}{p \cosh qL}. \tag{7–248}$$

Now, expanding the numerator and the denominator of the second term on the right-hand side of Eq. (7–248) in powers of q (or p), we have

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{1 + q^2x^2/2! + q^4x^4/4! + \cdots}{p(1 + q^2L^2/2! + q^4L^4/4! + \cdots)}. \tag{7–363}$$

Equation (7–363) may be rearranged in the usual manner to give

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{2a}(L^2 - x^2) - \frac{p}{4a^2}(L^2 - x^2)[L^2 - \tfrac{1}{6}(L^2 + x^2)] + \cdots \tag{7–364}$$

Since all terms of Eq. (7–364) are analytic, the inversion theorem yields readily $[T(x, t) - T_\infty]/(T_0 - T_\infty) = 0$, which again is the steady solution of the problem.

In fact, term-by-term inversion of the expansion of the transformed solution in powers of p gives the asymptote† rather than the asymptotic behavior for

† Some problems do not have a steady solution. In this case the asymptote implies the form of the solution as $t \rightarrow \infty$. See Example 7–13.

all lumped problems, and for all distributed problems that are solvable by Contour I of the inversion theorem. Remember that Contour I applies to problems whose transforms do not possess any branch points, that is, to problems whose geometry is finite in their orthogonal directions; these problems are also solvable by the separation of variables. However, for distributed problems whose transformed solutions do possess branch points, that is, for problems whose solutions may be obtained by Contour II of the inversion theorem,† the expansion of the transformed solution in powers of p and term-by-term inversion of the result gives the asymptotic behavior, rather than the asymptote. Let us illustrate this point by the following example.

Example 7–26. Let us find a solution of Example 7–3 valid for large values of time.

The solution of the transformed formulation of the problem may be taken from Example 7–3,

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{1}{p} - \frac{1}{p} e^{-qx}. \tag{7–51}$$

Expanding the second term on the right-hand side of Eq. (7–51) in powers of q, we obtain

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{x}{a^{1/2}p^{1/2}} - \frac{x^2}{a2!} + \frac{p^{1/2}x^3}{a^{3/2}3!} - \frac{px^4}{a^24!} + \frac{p^{3/2}x^5}{a^{5/2}5!} - \cdots \tag{7–365}$$

Eliminating its analytic terms, we may readily find the inversion of the remainder of Eq. (7–365) by TP(4), TP(8), and TP(9). The result is

$$\frac{T(x, t) - T_\infty}{T_0 - T_\infty} = \frac{1}{\pi^{1/2}}\left[\left(\frac{x^2}{at}\right)^{1/2} - \frac{1}{2 \cdot 3!}\left(\frac{x^2}{a}\right)^{3/2} + \frac{3}{4 \cdot 5!}\left(\frac{x^2}{at}\right)^{5/2} - \cdots\right]. \tag{7–366}$$

Thus, by contrast with the first two examples, we obtained the asymptotic behavior rather than the asymptote of our problem for large values of time.

We may now generalize the results of the foregoing three problems. Let the solution of the transformed formulation of a problem, say $T(\mathbf{r}, t)$, be $\overline{T}(\mathbf{r}, p)$, $\mathbf{r}$ being the position vector. If $\overline{T}(\mathbf{r}, p)$ has a number of singularities, an asymptotic expansion of $T(\mathbf{r}, t)$ can be deduced from the behavior of $\overline{T}(\mathbf{r}, p)$ near its singularity with largest real part.‡ Suppose this singularity to be at $p = \gamma_0$,

† It should be remembered that these problems cannot be solved by the separation of variables and the Fourier series (or the functions orthogonal over a finite interval) of Chapters 4 and 5. Only a few simple cases may be solved by these methods, and the procedure is somewhat elaborate even for simple problems. See, for example, Example 7–8.

‡ Note that relative to the possible roots on the negative half (most probably on the negative real axis) of the complex plane, $p = 0$ is the singularity with largest real part involved in the series expansions of the transforms of the last three examples.

where $\gamma_0 < \gamma$.† Then assume that $\overline{T}(\mathbf{r}, p)$ can be expanded near $p = \gamma_0$ in the form

$$\overline{T}(\mathbf{r}, p) = \sum_{n=0}^{\infty} a_n(\mathbf{r})(p - \gamma_0)^{n-1} + (p - \gamma_0)^{\beta-1} \sum_{n=0}^{\infty} b_n(\mathbf{r})(p - \gamma_0)^n, \tag{7–367}$$

where $0 < \beta < 1$. All terms of the first series of Eq. (7–367) except the first term are analytic and do not contribute to the solution. The inversion of the first term of the first series gives the asymptote of the problem. This asymptote plus the inversion of the second series and its rearrangement by the well-known property of the Gamma functions $\Gamma(\nu)\Gamma(1 - \nu) = \pi/\sin \pi\nu$, where ν may be a noninteger, give the asymptotic behavior of $T(\mathbf{r}, t)$ for large values of time in the form

$$T(\mathbf{r}, t) = e^{\gamma_0 t}\left[a_0(\mathbf{r}) + \frac{\sin \pi\beta}{\pi} \sum_{n=0}^{\infty} (-1)^n b_n(\mathbf{r})\Gamma(n + \beta)t^{-(n+\beta)}\right]. \tag{7–368}$$

If there are several singularities with the same real part, there must be a term of the type found in Eq. (7–368) corresponding to each of them.

The most frequent special case of Eq. (7–367) corresponds to $\gamma_0 = 0$, for which Eq. (7–367) becomes

$$\overline{T}(\mathbf{r}, p) = \sum_{n=0}^{\infty} a_n(\mathbf{r})p^{n-1} + p^{\beta-1} \sum_{n=0}^{\infty} b_n(\mathbf{r})p^n, \tag{7–369}$$

and then Eq. (7–368) yields

$$T(\mathbf{r}, t) = a_0(\mathbf{r}) + \frac{\sin \pi\beta}{\pi} \sum_{n=0}^{\infty} (-1)^n b_n(\mathbf{r})\Gamma(n + \beta)t^{-(n+\beta)}. \tag{7–370}$$

Furthermore, if $\beta = \frac{1}{2}$, Eq. (7–369) is reduced to

$$\overline{T}(\mathbf{r}, p) = \sum_{n=0}^{\infty} a_n(\mathbf{r})p^{n-1} + \sum_{n=0}^{\infty} b_n(\mathbf{r})p^{n-1/2}, \tag{7–371}$$

and Eq. (7–370) gives

$$T(\mathbf{r}, t) = a_0(\mathbf{r}) + \frac{1}{\pi^{1/2}} \sum_{n=0}^{\infty} (-1)^n b_n(\mathbf{r})(n - \tfrac{1}{2})(n - \tfrac{3}{2}) \cdots \tfrac{3}{2} \cdot \tfrac{1}{2} t^{-(n+1/2)}. \tag{7–372}$$

Since the second series of Eq. (7–369) is zero for the first two examples, only the asymptotes of these examples are obtained rather than their asymptotic

† See Fig. 7–20 for the definition of γ.

behaviors. On the other hand, Eq. (7–365) of the third example may be rearranged in the form of Eq. (7–371), which implies the existence of a solution giving the asymptotic behavior of the problem.

3. *Periodic solutions.* In Chapter 5 our attempt to solve problems subject to periodic disturbances by the separation of variables led us to a somewhat involved procedure. Despite the complexity of the procedure, we were able to obtain only the steady part of the periodic solution of a simple problem such as Example 5–6. Then in Chapter 6 we learned a convenient procedure for getting the steady solution of periodic problems, namely, the method of complex temperature; however, the complete solution of periodic problems has not yet been treated. Here we discuss the complete solution of these problems by the inversion theorem, and interpret their steady (periodic) part by means of their asymptotic behavior. Since no new concept is needed, these problems may best be explained by a couple of illustrative examples. First, however, let us note that a term (residue) contributed to the solution of a problem by a *simple* pole may be interpreted in physical terms by considering the location of this pole on the complex plane. The residue evaluated at a pole by Eq. (7–229) may depend on time only through the exponential term e^{tz} of the inversion theorem; therefore, interpreting e^{tz} by the location of a pole, we may draw the following conclusions when a problem under consideration has a simple pole: (a) at the origin, the contribution of this pole to the solution of the problem is a steady term; (b) on the negative real axis, the contribution is an exponentially decaying term; (c) on the imaginary axis, the contribution is a periodic term with constant amplitude; (d) on the negative half containing parts both on the imaginary and the negative real axes, the contribution is a periodic term with decaying amplitude.

Example 7–27. Let us find the complete periodic solution of the problem of Example 5–1 by the use of the inversion theorem.

The formulation of the problem is

$$\frac{d\theta}{dt} + m\theta = n \cos \omega t, \tag{5–1}$$

$$\theta(0) = 0. \tag{5–2}$$

The solution of the transformed formulation may readily be obtained in the form

$$\bar{\theta}(p) = \frac{np}{(p + m)(p^2 + \omega^2)}. \tag{7–373}$$

Inserting Eq. (7–373) into the inversion theorem, we find that the solution is

$$\frac{\theta(t)}{n} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ze^{tz}\, dz}{(z + m)(z^2 + \omega^2)}. \tag{7–374}$$

The integrand of Eq. (7–374) has simple poles at $z = -m$ and at $z = \pm i\omega$. Thus from Contour I we have

$$\frac{\theta(t)}{n} = \text{Res}\,(-m) + \text{Res}\,(\pm i\omega). \tag{7–375}$$

The residue at $z = -m$ is

$$\text{Res}\,(-m) = -\frac{me^{-mt}}{(m^2 + \omega^2)},$$

and the residues at $z = \pm i\omega$ are

$$\text{Res}\,(i\omega) = \frac{e^{i\omega t}}{2(m + i\omega)}, \qquad \text{Res}\,(-i\omega) = \frac{e^{-i\omega t}}{2(m - i\omega)}.$$

Using these results in Eq. (7–375), we obtain the previously found solution

$$\frac{T(t) - T_\infty}{n/m} = \frac{m}{(m^2 + \omega^2)^{1/2}} \cos(\omega t - \alpha) - \frac{m^2 e^{-mt}}{(m^2 + \omega^2)}, \quad \alpha = \tan^{-1}(\omega/m). \tag{5–3}$$

As we have learned, the asymptotic behavior of Eq. (5–3) may be deduced from the behavior of Eq. (7–373) near its singularities with the largest real part, that is, near the two poles on the imaginary axis, $z = \pm i\omega$ (Fig. 7–26). Thus the first term on the right-hand side of Eq. (5–3), corresponding to the sum of the residues at $z = \pm i\omega$ only, is periodic with a constant amplitude, that is, it constitutes the steady part of the complete periodic solution.

Finally, let us consider the complete periodic solution of Example 5–6, which has been avoided so far.

Example 7–28. Let us find the complete periodic solution of Example 5–6 by the inversion theorem.

The formulation of the problem may be taken from Example 5–6. Thus

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}, \tag{5–35}$$

$$\theta(x, 0) = 0, \tag{5–36}$$

$$\theta(0, t) = \theta_0 \cos \omega t, \tag{5–37}$$

$$\theta(\infty, t) = 0. \tag{5–38}$$

The solution of the transformed formulation is

$$\frac{\bar{\theta}(x, p)}{\theta_0} = \frac{pe^{-qx}}{(p^2 + \omega^2)}, \tag{7–376}$$

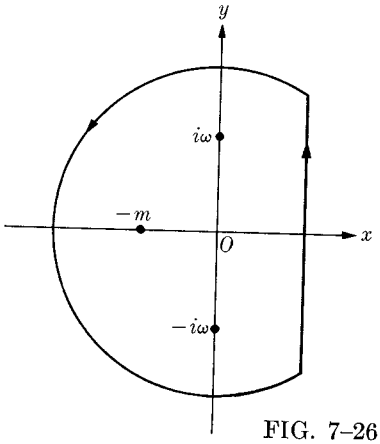


FIG. 7–26

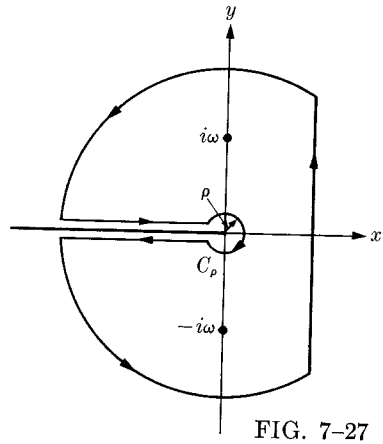


FIG. 7–27

and the inversion theorem gives

$$\frac{\theta(x,t)}{\theta_0} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{ze^{tz-(z/a)^{1/2}x}}{(z^2+\omega^2)}\,dz. \tag{7–377}$$

The integrand of Eq. (7–377) has a branch point at $z = 0$ and two simple poles at $z = \pm i\omega$ (Fig. 7–27). Thus from Contour II and Eq. (7–236) we have

$$\frac{\theta(x,t)}{\theta_0} = \text{Res}\,(\pm i\omega) + \lim_{\rho\to 0}\int_{C_\rho} \frac{ze^{tz-(z/a)^{1/2}x}}{(z^2+\omega^2)}\,dz - \frac{1}{2\pi i}\int_0^\infty \frac{\rho e^{-t\rho}[e^{i(\rho/a)^{1/2}x} - e^{-i(\rho/a)^{1/2}x}]}{(\rho^2+\omega^2)}\,d\rho. \tag{7–378}$$

The residues at $z = \pm i\omega$ are

$$\text{Res}\,(i\omega) = \tfrac{1}{2}e^{i\omega t-(1+i)(\omega/2a)^{1/2}x}, \qquad \text{Res}\,(-i\omega) = \tfrac{1}{2}e^{-i\omega t-(1-i)(\omega/2a)^{1/2}x}.$$

The sum of these residues,

$$\text{Res}\,(i\omega) + \text{Res}\,(-i\omega) = e^{-(\omega/2a)^{1/2}x}\cos\,[\omega t - (\omega/2a)^{1/2}x], \tag{5–44}$$

is the steady part of the solution. Since its integrand approaches zero as $\rho(\text{or } z) \to 0$, the integral around the small circle C_ρ vanishes.

Employing these results in Eq. (7–378), and expressing the last integral of this equation in terms of a circular function, we obtain finally

$$\frac{\theta(x,t)}{\theta_0} = e^{-(\omega/2a)^{1/2}x}\cos\,[\omega t - (\omega/2a)^{1/2}x] - \frac{1}{\pi}\int_0^\infty \frac{\rho e^{-t\rho}\sin\,(\rho/a)^{1/2}x}{(\rho^2+\omega^2)}\,d\rho. \tag{7–379}$$

The integral involved in Eq. (7–379) cannot be integrated explicitly; a numerical integration procedure may be employed for its parametric study.

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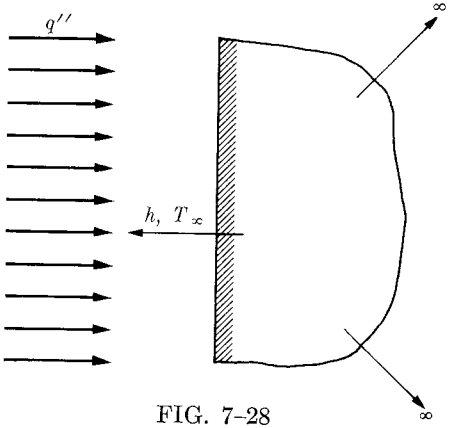


FIG. 7–28

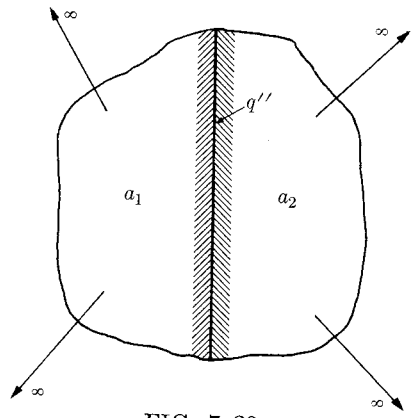


FIG. 7–29

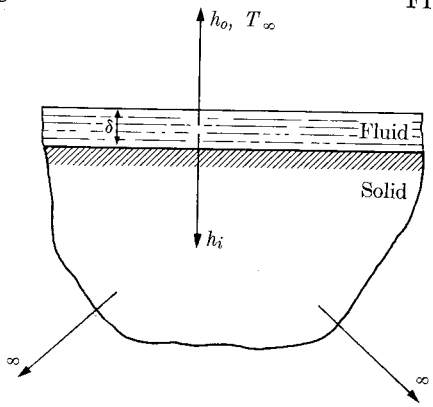


FIG. 7–30

Problems

7–1. A semi-infinite solid is suddenly subjected to the radiant heat flux q'' acting at a distance (Fig. 7–28). Initially the solid is at the ambient temperature. The heat transfer coefficient between the solid and the ambient is h. Find the unsteady temperature in the system.

7–2. Find the unsteady temperature in the problem of Example 3–9 corresponding to a sudden change in the base temperature from 0 to θ_0.

7–3. An infinite medium is composed of two semi-infinite solids having different diffusivities, say a_1 and a_2 (Fig. 7–29). The initial temperature of the system is uniform; then the interface suddenly assumes the heat flux q'', which may result from an electrically heated plate of negligible thickness. Find the unsteady temperature in the solids.

7–4. A fluid at temperature T_0 is suddenly poured over a semi-infinite solid at temperature T_∞. Assume that the fluid instantaneously becomes a thin layer of thickness δ covering the entire surface of the solid (Fig. 7–30). The heat transfer coefficient between the fluid and the ambient is h_o. Find the unsteady temperature in the system, assuming that the heat transfer coefficient h_i between the fluid and the solid is (a) infinite, (b) finite.

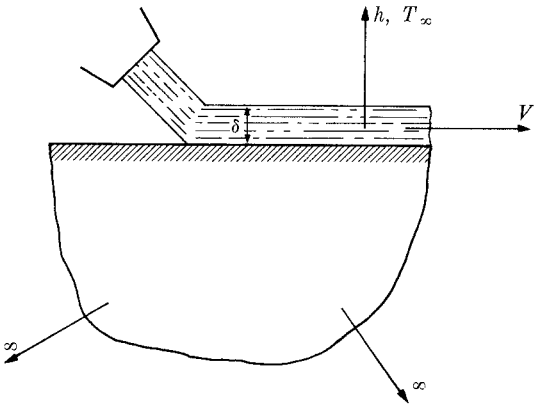


FIG. 7–31

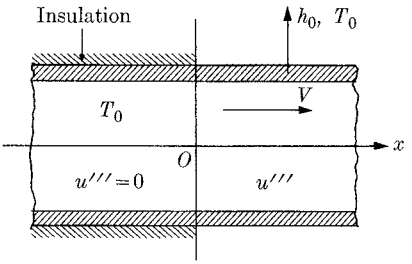


FIG. 7–32

7–5. A thin layer of fluid having the thickness δ, velocity V, and temperature T_0 attaches to a semi-infinite solid at a point, and thereafter flows over the solid (Fig. 7–31). The initial temperature of the solid is T_∞. Let h denote the heat transfer coefficient between the fluid and the ambient, and assume that the heat transfer coefficient between the liquid and the solid is large. Conduction is negligible in the direction of flow, both in the fluid and in the solid. Find the unsteady temperature in the system.

7–6. A nuclear fluid at temperature T_0 flows steadily through an infinitely long tube (Fig. 7–32). The upstream half of the tube is insulated, while the downstream half may transfer heat with a heat transfer coefficient h_0 to an ambient at temperature T_0. Assume now that constant nuclear energy u''' is uniformly generated in the downstream half of the fluid. Axial conduction is negligible both in the fluid and in the tube walls. Find the unsteady temperatures in the system, on the basis of a radially lumped and axially distributed analysis.

7–7. Re-solve Example 7–7 in terms of an integral formulation and appropriately selected approximate temperature profiles.

7–8. Re-solve the insulated case of Example 7–5, assuming that the inlet temperature T_0 of the liquid is changing as shown in Fig. 7–33, rather than being changed suddenly.

7–9. A cross-flow heat exchanger may be simulated by three parallel plates of negligible thicknesses. The upper and lower surfaces of the exchanger are insulated. Assume that the fluids flow steadily with the velocities U_1 and U_2 (Fig. 7–34), that the inlet temperatures of the fluids, T_{1_0} and T_{2_0}, are uniform, and that axial conduction is negligible. Find the *steady* temperature in the system on the basis of a transversally lumped but otherwise differential analysis.

7–10. The unsteady thermal behavior of closed containers during discharge of the container fluid by a pressurant gas finds important application in current problems of space technology. The following analytical model is suggested for the investigation of this problem (Fig. 7–35). (i) The top and the bottom of the container are insulated. (ii) Axial conduction in the system is negligible. (iii) The physical properties are assumed to be constant. (iv) The time dependence of the pressurant gas velocity is

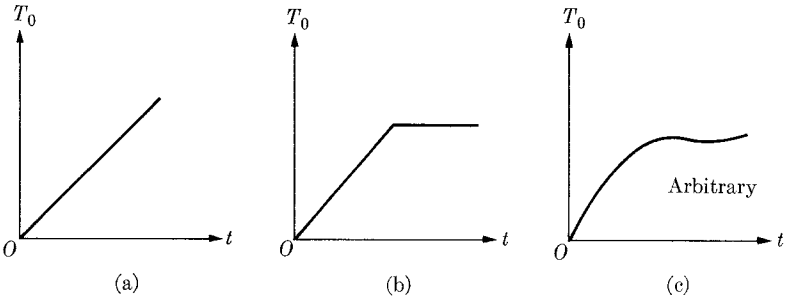


FIG. 7–33

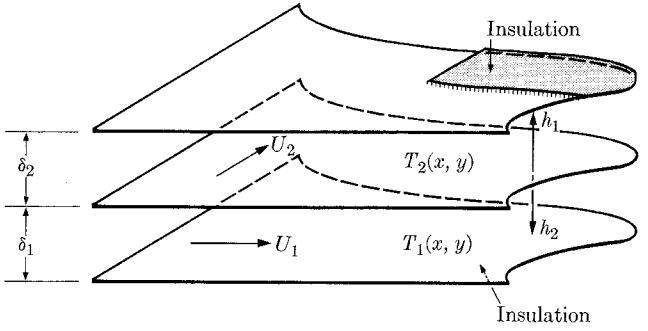


FIG. 7–34

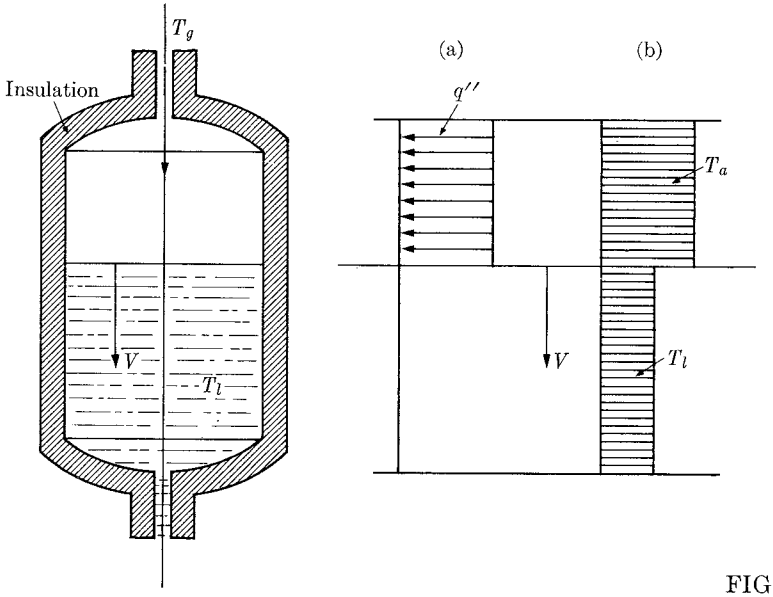


FIG. 7–35

negligible compared to that of its temperature. (v) The heat transfer between the fluid and the ambient is small compared to that between the pressurant gas and the ambient.† (vi) The periphery of the container is subjected to a uniform heat flux q'', or transfers heat to an ambient at temperature T_a. Since the heat transfer below the gas-fluid interface is negligible according to (v), q'' and T_a may be assumed to be moving with the interface velocity as shown in Fig. 7–35. (vii) Initially the container is filled with a cryogenic fluid at temperature T_l. Then the fluid is drained out of the container with constant velocity V by a pressurized gas having the inlet temperature T_g. Find the unsteady temperatures in the system on the basis of a radially lumped and axially distributed analysis.

7–11. Two fluids flow steadily in a parallel-flow heat exchanger (Fig. 7–36). The inlet temperature and the velocity of the fluids are T_i, θ_i and U, V, respectively. The tube walls which separate the fluids have negligible heat capacity.‡ The outer surface of the heat exchanger is insulated. (a) Find the steady temperatures in the system on the basis of a radially lumped and axially distributed analysis. (b) Assume that the inlet temperature of one fluid is suddenly changed to another temperature, say from T_i to T_i'. Find the unsteady temperatures in the system.

7–12. Re-solve Problem 7–11 for (i) an infinitely long, (ii) a finite counterflow heat exchanger.

7–13. Re-solve Example 3–10 by the use of Laplace transforms. The following procedure is suggested. Starting from the formulation of the problem,

$$d^2\theta/dx^2 - m^2\theta = 0, \tag{3–163}$$

$$d\theta(0)/dx = 0, \tag{3–170}$$

$$\theta(L) = \theta_0, \tag{3–171}$$

integrate Eq. (3–163) twice over the interval $(0, x)$ to get

$$\theta(x) = \theta(0) + m^2\int_0^x \int_0^x \theta(\xi)\, d\xi\, d\xi, \tag{7–380}$$

where ξ is the dummy variable; using Leibnitz's integral formula,§ show from the integral

$$I = \int_0^x (x - \xi)\theta(\xi)\, d\xi$$

that

$$\int_0^x \int_0^x \theta(\xi)\, d\xi\, d\xi = \int_0^x (x - \xi)\theta(\xi)\, d\xi; \tag{7–381}$$

then rearrange Eq. (7–380) by considering Eq. (7–381); take the Laplace transforms of the result in x; eliminate $\theta(0)$ from the solution by using Eq. (3–171).

7–14. Find the *unsteady* behavior of the system of Problem 7–9 corresponding to a sudden change in the inlet temperature of one fluid. [*Hint:* Use Laplace trans-

† See Reference 18.

‡ See Problem 5–5 for the formulation of the problem corresponding to the case of counterflow and including the heat capacity of the tube walls.

§ See Problem 2–8.

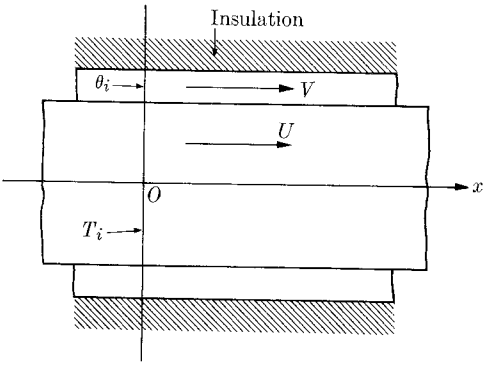
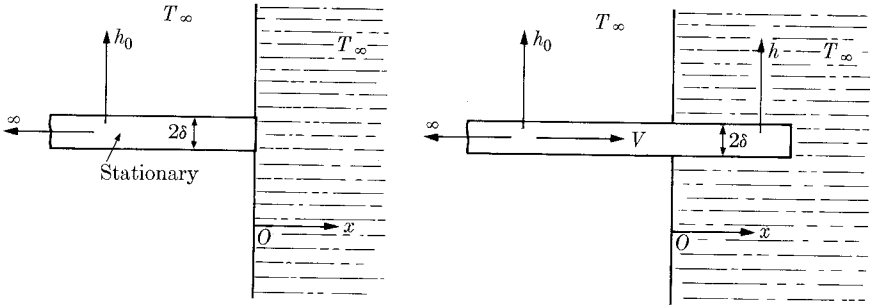


FIG. 7–36



Initially FIG. 7–37 Any time

forms successively in the time and the space in which the inlet temperature of the fluid is kept unchanged.]

7–15. Consider a semi-infinite ($x < 0$) metal sheet of thickness 2δ. The *instantaneous* initial temperature of the sheet is T_0; then the sheet suddenly assumes a uniform velocity V and moves into the second half ($x > 0$) of an infinite domain (Fig. 7–37). The ambient temperature of the first and second half-domains is the same, say T_∞. The second half-domain is a liquid metal bath. The momentum (velocity) of the moving sheet penetrates to a small depth in the liquid metal. This depth may be neglected for the intended investigation, and the entire second half-domain may be assumed stationary. *Axial conduction is negligible.* Find the unsteady temperature in the second half-domain corresponding to (i) an infinite heat transfer coefficient h, (ii) a finite heat transfer coefficient h.

7–16. Re-solve Problem 5–9.

7–17. Re-solve Problem 5–10.

7–18. Re-solve Problem 5–11.

7–19. Re-solve Problem 5–12.

7–20. The outside walls of a house are assumed to be made of composite walls consisting of a flat plate of thickness L and a semi-infinite body (Fig. 7–38). On a cloudy summer day the inside and the outside of the house may be assumed to have

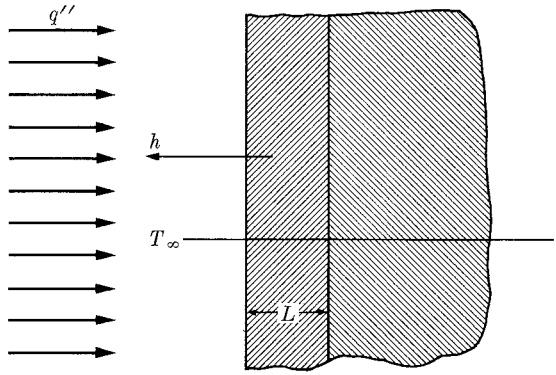


FIG. 7–38

the same temperature, T_∞. During the day the sun suddenly starts to shine and imposes the uniform heat flux q'' on the walls for a period of time. Find the unsteady temperature in the outside walls.

7–21. Re-solve Problem 5–15.

7–22. Re-solve Problem 5–16.

7–23. Re-solve Problem 5–17.

7–24. Re-solve Example 7–19, assuming that the tube is not insulated and that it transfers heat with a heat transfer coefficient h_0 to the ambient at temperature T_∞.

7–25. Re-solve Problem 5–18.

7–26. The process of drilling rocks may be simulated by an infinite solid having a cylindrical hole that contains an infinitely long drill (Fig. 7–39). Thus the axial motion of the drill is eliminated from the formulation of the problem. The initial temperature of the system is uniform, say T_∞; then the drill starts drawing a constant power from the power supply. It may be assumed that the total power drawn by the drill is uniformly transferred to the solid. Find the unsteady temperature in the solid.

7–27. Re-solve the preceding problem, assuming that the drill is a solid rod and that only a part of the total power drawn from the power supply is (uniformly) transferred to the solid.

7–28. Re-solve Problem 5–23.

7–29. Re-solve Problem 5–24.

7–30. Assume that the radiant energy generated in a spherical hole in an infinite solid may be considered as a sudden heat flux imposed on the surface of the hole (Fig. 7–40). The initial temperature of the solid is uniform, say T_∞. Find the unsteady temperature in the solid.

7–31. Re-solve Problem 5–27.

7–32. Re-solve Example 7–21 by means of the Laplace transforms successively considered in time and in the space (direction) in which the geometry of the problem extends to infinity. [*Hint:* First establish the property

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = p^2\bar{f}(p) - pf(0+) - \frac{df(0+)}{dt}$$

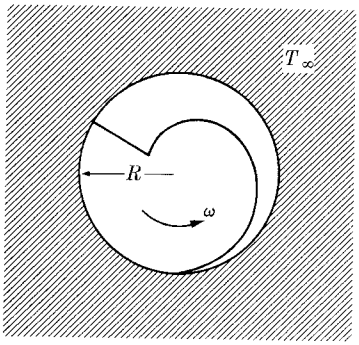


FIG. 7-39

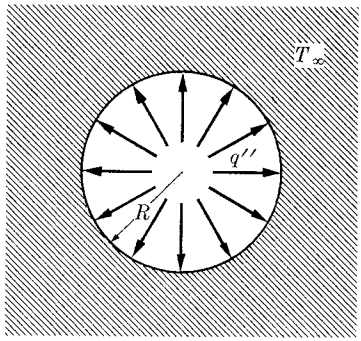


FIG. 7-40

in a manner analogous to the establishment of TP(II). Recall that t does not necessarily imply time.]

7–33. Solve the unsteady problem associated with Example 4–21, assuming that the initial temperature of the rod is zero and that the base temperature is suddenly changed to θ_0.

7–34. Re-solve Example 5–15.

7–35. Are there any solutions of Problem 7–1 through Problem 7–5 useful for small or large values of time? First state your answers for each problem. Next consider at least one of these problems and, employing the method of solution suitable to small or large values of time, back up your answers.

7–36. Repeat Problem 7–35 for Problems 7–6 through 7–12 except Problems 7–7 and 7–9.

7–37. Repeat Problem 7–35 for Problems 7–15 through 7–20.

7–38. Find a solution of Example 7–25 for small values of time and small values of r/R. [*Hint:* Use the small-argument expansion of the numerator of the transformed solution.]

7–39. Repeat Problem 7–35 for Problems 7–21 through 7–27.

7–40. Repeat Problem 7–35 for Problems 7–28 through 7–34.

7–41. Find solutions of Problem 4–25 useful for small or large values of time corresponding to a sudden change in the internal energy generation in the rod from 0 to u'''.†

7–42. Find the transient periodic solution of Problem 6–6.

7–43. Find the transient periodic solution of Example 6–4.

7–44. Find the transient periodic solution of Problem 6–7.

7–45. Find the transient periodic solution of Problem 6–9.

7–46. Find the transient periodic solution of Example 6–3.

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† See Reference 20.

PART III | FURTHER METHODS OF FORMULATION AND SOLUTION

VARIATIONAL FORMULATION. SOLUTION BY APPROXIMATE PROFILES

In Chapter 2 we classified the formulations of problems as *lumped* and *distributed* (integral, differential, variational, and difference). So far we have discussed the lumped, integral, and differential formulations, and the solution methods suitable to each. Since both the variational and difference formulations can be obtained from the differential formulation by means of mathematics, we have postponed the study of these formulations until the necessary mathematical foundation could be laid. This chapter is devoted to variational formulation and solution, and Chapters 9 and 10 to difference formulation and solution.

Here we first introduce the concept of variational calculus and briefly develop its properties; then we shall apply the results to some problems. Also, we shall discuss at length the selection of profiles, which until now has been loosely treated.

8–1. Basic Problem of Variational Calculus

An important problem of differential calculus is to find the extreme (stationary) values of a function. Given, for example, a function of one independent variable, $y = y(x)$, the necessary condition for the existence of extreme values of this function is $dy/dx = 0$. The sufficient condition for the function to be minimum or maximum is $d^2y/dx^2 > 0$ or $d^2y/dx^2 < 0$, respectively, at the point where $dy/dx = 0$. Variational calculus owes its origin to a similar but more complicated problem, that of finding a function $y(x)$ such that a definite integral whose integrand is a function of this function shall assume an extreme value. The necessary condition for the existence of an extreme value of the integral is the subject matter of this chapter. (The sufficient condition for the extreme value to correspond to a minimum or a maximum may be obvious for a given physical problem, but it is rather difficult to prove and will not be considered here.)

As an introduction, we present one of the classical examples of variational calculus, a problem which readily permits physical interpretation of the extremum being sought; let us find the equation of the shortest plane curve joining two given points (Fig. 8–1). The differential length of a curve $y = y(x)$ connecting these points is

$$ds = (dx^2 + dy^2)^{1/2},$$

which can be rearranged as

$$ds = (1 + y'^2)^{1/2}.$$

The total length of this curve,

$$S = \int_a^b (1 + y'^2)^{1/2}\, dx, \qquad (8\text{–}1)$$

is now required to be a minimum.

Before proceeding with the solution of this problem, let us consider the general problem of finding the extreme values of the integral

$$I = \int_a^b F(x, y, y')\, dx, \qquad (8\text{–}2)$$

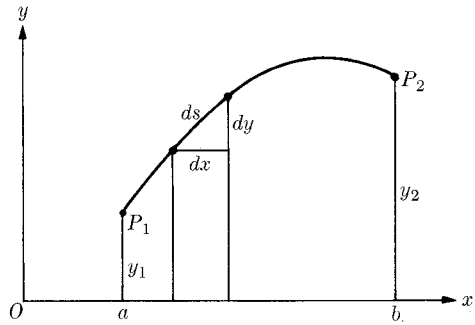


FIG. 8–1

where $F(x, y, y')$ is a function known from the statement of the problem, while y is the unknown to be determined such that the integral becomes stationary. A function defined by a definite integral whose integrand itself depends on functions is said to be a *functional*. The integrand of this integral may also be called a functional when the independent variable x is considered fixed but the dependent variable y is varied. The integral of Eq. (8–2) takes on different values along different paths connecting the fixed points P_1, P_2. We wish to find the particular curve $y(x)$ which makes the integral, for example, a minimum.

If it is assumed that $y(x)$ actually minimizes this integral, any function in the neighborhood of $y(x)$ can be represented by the form $y(x) + \epsilon\eta(x)$, where $\eta(x)$ is a continuously differentiable function that vanishes at the ends of the interval, and ϵ is a parameter (Fig. 8–2). Then, in terms of these functions, the integral of Eq. (8–2) can be written in the form

$$I(\epsilon) = \int_a^b F(x, y + \epsilon\eta, y' + \epsilon\eta')\, dx, \qquad (8\text{–}3)$$

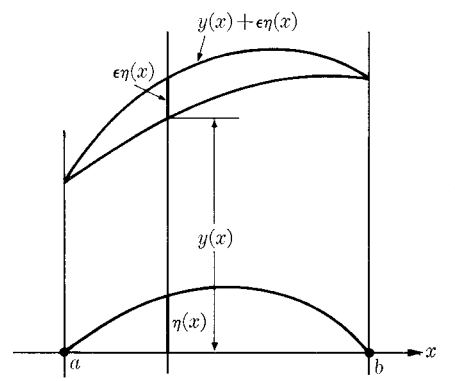


FIG. 8–2

which, according to the assumption about $y(x)$, assumes its minimum value when $\epsilon = 0$.

Since in the (I, ϵ)-plane $I(\epsilon)$ has a minimum for $\epsilon = 0$, according to differential calculus $I(\epsilon)$ satisfies the necessary condition for extrema at this point, namely

$$dI(\epsilon)/d\epsilon = 0 \qquad \text{when} \qquad \epsilon = 0. \qquad (8\text{–}4)$$

Now, defining

$$F(\epsilon) = F(x, y + \epsilon\eta, y' + \epsilon\eta'), \qquad (8\text{–}5)$$

and *assuming* that the order of integration and differentiation can be interchanged, we combine Eqs. (8–3), (8–4), and (8–5) to give

$$\frac{dI(\epsilon)}{d\epsilon} = \int_a^b \frac{dF(\epsilon)}{d\epsilon}\, dx. \tag{8–6}$$

Since for a fixed value of x the derivative of Eq. (8–5) with respect to ϵ is

$$\frac{dF(\epsilon)}{d\epsilon} = \frac{\partial F(\epsilon)}{\partial(y + \epsilon\eta)}\,\eta + \frac{\partial F(\epsilon)}{\partial(y' + \epsilon\eta')}\,\eta', \tag{8–7}$$

noting that $F(\epsilon)|_{\epsilon=0} = F$ and using Eqs. (8–3), (8–4), (8–5), (8–6), and (8–7), we may rearrange Eq. (8–4) in the form

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \left(\frac{\partial F}{\partial y}\,\eta + \frac{\partial F}{\partial y'}\,\eta'\right) dx = 0.$$

Integrating the second term of this integral by parts, we obtain

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\, dx + \left.\frac{\partial F}{\partial y'}\,\eta\right|_a^b = 0. \tag{8–8}$$

Since, by definition, η vanishes at the ends of the interval, the second term of Eq. (8–8) is zero. Hence

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\, dx = 0. \tag{8–9}$$

The integral of Eq. (8–9) vanishes for any $\eta(x)$ when

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{8–10}$$

Equation (8–10) is the *Euler equation* associated with the variational problem given by Eq. (8–2). Thus *the condition necessary for $y(x)$ to minimize (or maximize) the integral of Eq.* (8–2) *is that $F(x, y, y')$ must satisfy the corresponding Euler equation.* Note that the partial derivatives $\partial F/\partial y$ and $\partial F/\partial y'$ of Eq. (8–10) are formed by assuming y and y' as independent variables; this is necessary since the variation is on these, not on the independent variable x.

Let us now return to our specific problem, the plane curve which gives the shortest distance between two points. Using Eq. (8–1) and noting that $F = (1 + y'^2)^{1/2}$, we find that the Euler equation of the problem is

$$\frac{d}{dx}\left[\frac{y'}{(1 + y'^2)^{1/2}}\right] = 0,$$

which integrates to

$$y'/(1 + y'^2)^{1/2} = C \qquad \text{or} \qquad y' = C_1.$$

The next integration gives

$$y = C_1 x + C_2,$$

where C, C_1, and C_2 are constants. Thus we are led to the well-known fact that the shortest plane curve joining two points is a straight line.

The general problem discussed above may be extended by assuming that $y(x)$ is fixed at one end, say at $x = a$, and is arbitrary at $x = b$. Now, defining $\eta(x)$ such that it is continuously differentiable yet is zero at $x = a$ only, we may write Eq. (8–8) in the form

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta \, dx + \frac{\partial F}{\partial y'} \eta \Big|_{x=b} = 0. \tag{8–11}$$

One way to satisfy Eq. (8–11) is to let each of its two terms separately be equal to zero. The first term then gives the previous result, the Euler equation, while the second term, since $\eta(b)$ is arbitrary, introduces the additional condition for the new problem,

$$\frac{\partial F}{\partial y'}\Big|_{x=b} = 0, \tag{8–12}$$

as a constraint on the Euler equation. Clearly, a further extension would involve $y(x)$ arbitrary at both ends.

8–2. Meaning and Rules of Variational Calculus

The differential of a function is a first-order approximation to the change in the value of this function along the particular curve it describes, while the variation of the same function is a first-order approximation to the change in this function from the curve it describes to another curve by virtue of a contained parameter (Fig. 8–3). The change $\epsilon\eta(x)$ in $y(x)$ is called the *variation of* y and is denoted by

$$\delta y = \epsilon\eta. \tag{8–13}$$

Now, corresponding to the variation of y, the change in the functional $F(x, y, y')$ for a fixed x is

$$\Delta F = F(x, y + \epsilon\eta, y' + \epsilon\eta') - F(x, y, y'). \tag{8–14}$$

Assuming y and y' as independent variables, and expanding the right-hand side of Eq. (8–14) into a two-dimensional Taylor series around (y, y'), we obtain

$$\Delta F = \frac{\partial F}{\partial y} \epsilon\eta + \frac{\partial F}{\partial y'} \epsilon\eta' + O(\epsilon^2). \tag{8–15}$$

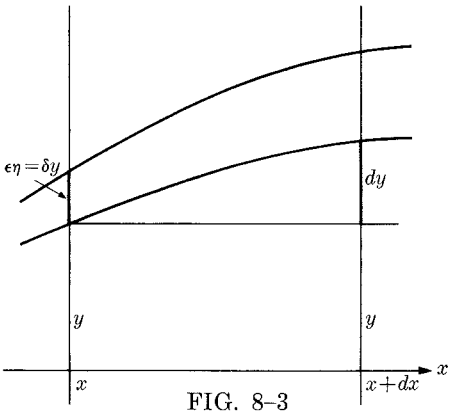


FIG. 8–3

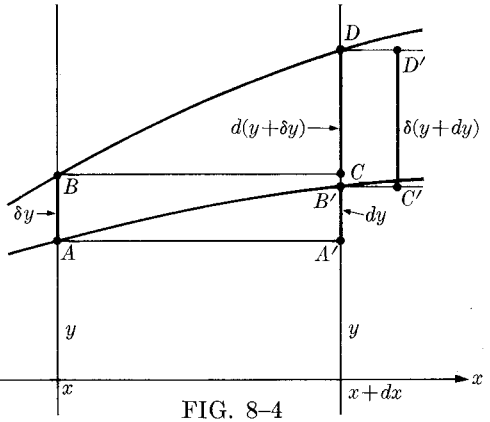


FIG. 8–4

In a manner analogous to the definition of the differential of a function, the first two terms of the right-hand member of Eq. (8–15) are defined to be the variation of the functional $F(x, y, y')$ which, with the use of Eq. (8–13), may be written in the form

$$\delta F = \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y'. \tag{8–16}$$

Noting that $\delta x = 0$, and thus adding the identically zero term $(\partial F/\partial x)\,\delta x$ to Eq. (8–16), we clearly see the analogy between the definitions of the differential of a function and the variation of a functional.

From the definition of variation, it may be shown that the rules for the variation of sums, products, ratios, powers, and so on, are completely analogous to the corresponding rules of differentiation.

The commutative property, that *the differential of the variation of a function $y(x)$ is identical to the variation of the differential of the same function,* is one of the most important properties of the variational calculus. This statement may readily be proved by means of Fig. 8–4 as follows:

$$\begin{aligned} AB + CD &= A'B' + C'D' \\ \delta y + d(y + \delta y) &= dy + \delta(y + dy) \\ d(\delta y) &= \delta(dy). \end{aligned} \tag{8–17}$$

Finally, summing up our general discussion of one-dimensional variational problems, we shall show that *the necessary condition for the existence of an extremum of the functional*

$$I = \int_a^b F(x, y, y')\,dx \tag{8–18}$$

is the vanishing of its variation,

$$\delta I = \delta \int_a^b F(x, y, y')\,dx = 0 \tag{8–19}$$

provided

$$\frac{\partial F}{\partial y'}\,\delta y\Big|_a^b = 0 \tag{8–20}$$

*is satisfied.**

Since variation and integration, like variation and differentiation, are commutative, we may now write Eq. (8–19) in the form

$$\delta I = \int_a^b \delta F(x, y, y')\,dx = 0$$

or, using Eq. (8–16),

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial y'}\,\delta y'\right) dx = 0. \tag{8–21}$$

Next, rearranging $\delta y'$ by means of Eq. (8–17), we obtain

$$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{\delta(dy)}{dx}$$
$$= \frac{d(\delta y)}{dx} = \frac{d}{dx}(\delta y),$$

and integrating the second term of Eq. (8–21) by parts gives

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx + \frac{\partial F}{\partial y'}\,\delta y\Big|_a^b = 0.$$

This equation is proportional to Eq. (8–8), ϵ being the proportionality constant, and with Eq. (8–20) reduces to

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y\,dx = 0, \tag{8–22}$$

or to the Euler equation given by Eq. (8–10), the condition necessary for I to be stationary.

8–3. Steady One-Dimensional Problems

On the basis of the preceding discussion, we may obtain the variational formulation of steady one-dimensional conduction problems. First, however, let us note a basic difference between the variational formulation of geometric and of

* Equation (8–20) is often referred to as the *natural boundary condition* of the variational problem.

physical problems. *The statement of geometric problems* (for example, the distance between two points, the surface of revolution passing through two points, etc.) *is the functional of their variational formulations. The use of the corresponding Euler's equation gives then the extreme values of this functional* (the shortest distance between two points, the smallest surface of revolution passing through two points, etc.). By contrast, *the statement of physical problems cannot explicitly be related to a functional to be extremized on the basis of a physical reasoning.* Reconsider the general problem discussed in connection with Eqs. (8–2) through (8–10). Conceptually speaking, what we have learned by this problem has been *the existence of a differential equation as the condition necessary for an appropriate integral to be stationary.* Since physical problems are, in general, expressible in terms of differential equations, we may obtain the variational formulation of these problems by following the steps of the variational formulation of geometric problems in the reverse order. Thus we start by considering the differential formulation of a physical problem to be the Euler equation associated with its desired variational formulation. The point will further be clarified by the following example.

Example 8–1. Let us find the variational formulation of Example (3–10). The differential formulation of the problem is

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0, \tag{3–163}$$

$$\frac{d\theta(0)}{dx} = 0, \tag{3–170}$$

$$\theta(L) = \theta_0. \tag{3–171}$$

Now, taking the governing differential equation of this problem, Eq. (3–163), as the Euler equation of the desired variational problem, we have from Eq. (8–22)

$$\delta I = \int_0^L \left(\frac{d^2\theta}{dx^2} - m^2\theta\right) \delta\theta \, dx = 0. \tag{8–23}$$

Integrating the first term of Eq. (8–23) by parts gives

$$\frac{d\theta}{dx}\,\delta\theta\Big|_0^L - \int_0^L \left(\frac{d\theta}{dx}\right) \frac{d}{dx}(\delta\theta)\, dx - m^2 \int_0^L \theta\, \delta\theta\, dx = 0. \tag{8–24}$$

Using the commutative property

$$\frac{d}{dx}(\delta\theta) = \delta\left(\frac{d\theta}{dx}\right)$$

for the middle term, and noting that

$$\left(\frac{d\theta}{dx}\right)\delta\left(\frac{d\theta}{dx}\right) = \tfrac{1}{2}\delta\left(\frac{d\theta}{dx}\right)^2, \qquad \theta\,\delta\theta = \tfrac{1}{2}\,\delta\theta^2,$$

we may rearrange Eq. (8–24) as follows:

$$\left(\frac{d\theta}{dx}\right)\delta\theta\Big|_0^L - \tfrac{1}{2}\delta\int_0^L\left[\left(\frac{d\theta}{dx}\right)^2 + m^2\theta^2\right]dx = 0. \tag{8–25}$$

The boundary conditions $\theta = \text{const}$ and $(d\theta/dx) = 0$ for which the first term of Eq. (8–25) vanishes are the natural boundary conditions of the problem. Thus the variational formulation of the problem is

$$\delta\int_0^L\left[\left(\frac{d\theta}{dx}\right)^2 + m^2\theta^2\right]dx = 0, \tag{8–26}$$

and the corresponding functional is

$$I = \int_0^L\left[\left(\frac{d\theta}{dx}\right)^2 + m^2\theta^2\right]dx. \tag{8–27}$$

Let us now reconsider the problem of Example 8–1, and keep the end insulated but replace the base temperature with the specified heat flux

$$+k\,\frac{d\theta(L)}{dx} = q''. \tag{8–28}$$

The variational formulation of Example 8–1 applies equally to the present case before the use of boundary conditions. Introducing Eqs. (3–170) and (8–28) into Eq. (8–25), we find that the variational form of the new problem is

$$\frac{q''}{k}\,\delta\theta(L) - \tfrac{1}{2}\,\delta\int_0^L\left[\left(\frac{d\theta}{dx}\right)^2 + m^2\theta^2\right]dx = 0$$

or

$$\delta\left\{\frac{q''}{k}\,\theta(L) - \tfrac{1}{2}\int_0^L\left[\left(\frac{d\theta}{dx}\right)^2 + m^2\theta^2\right]dx\right\} = 0. \tag{8–29}$$

Thus, *unlike the differential formulation, which involves the governing differential equation and the boundary conditions separately, the variational formulation includes the boundary conditions in the governing variational equation.* Therefore, *a problem may yield different variational formulations, depending on the boundary conditions.*

We may extend the foregoing cartesian problem to other geometries by considering a general form of linear second-order differential equations:*

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y = f(x). \tag{8–30}$$

Let us find the variational form of Eq. (8–30) over the interval (a, b). Taking Eq. (8–30) as the Euler equation of the desired variational problem, we have

$$\int_a^b \left[\frac{d}{dx}\left(p\frac{dy}{dx}\right) - qy - f\right]\delta y\, dx = 0.$$

Integrating the first term of this equation by parts gives

$$\left(p\frac{dy}{dx}\right)\delta y\Big|_a^b - \int_a^b \left[\left(p\frac{dy}{dx}\right)\frac{d}{dx}(\delta y) + qy\,\delta y + f\,\delta y\right]dx = 0,$$

which may be rearranged in the form

$$\left(p\frac{dy}{dx}\right)\delta y\Big|_a^b - \delta\int_a^b \left[\tfrac{1}{2}p\left(\frac{dy}{dx}\right)^2 + \tfrac{1}{2}qy^2 + fy\right]dx = 0.$$

If the boundary conditions are natural, the first term is zero, and the variational form of Eq. (8–30) is

$$\delta\int_a^b \left[\tfrac{1}{2}p\left(\frac{dy}{dx}\right)^2 + \tfrac{1}{2}qy^2 + fy\right]dx = 0. \tag{8–31}$$

If the boundary conditions are of the general type

$$\frac{dy(a)}{dx} + B_1 y(a) = 0, \qquad \frac{dy(b)}{dx} + B_2 y(b) = 0,$$

---

* A linear second-order differential equation written in the form

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = f_3(x)$$

may be reduced to Eq. (8–30) by the transformation:

$$p(x) = \exp\left[\int f_1(x)\,dx\right], \qquad q(x) = -f_2(x)p(x), \qquad \text{and} \qquad f(x) = f_3(x)p(x).$$

In Chapter 4 we considered a similar but homogeneous equation, Eq. (4–16), in connection with characteristic-value problems.

the variational form becomes

$$\delta\left\{-\tfrac{1}{2}B_2p(b)y^2(b) + \tfrac{1}{2}B_1p(a)y^2(a) - \int_a^b\left[\tfrac{1}{2}p\left(\frac{dy}{dx}\right)^2 + \tfrac{1}{2}qy^2 + fy\right]dx\right\} = 0. \tag{8–32}$$

Other forms corresponding to other boundary conditions may be obtained in a similar manner.

Although cartesian geometry requires no rearrangement, a linear second-order differential equation written in terms of cylindrical, spherical, or other coordinates cannot be transformed into a variational form unless it is first expressed in the form of Eq. (8–30). [Take, for example, cylindrical geometry and try the variational form of $r^2(d^2\theta/dr^2) + r(d\theta/dr)$ or $d^2\theta/dr^2 + (1/r)(d\theta/dr)$.]

Now that we have learned how to obtain the variational form of steady one-dimensional problems, we shall make use of this form to obtain approximate solutions to these problems by a procedure analogous to that used with the integral formulation, as follows. It is possible to construct a number of geometrical profiles which satisfy the prescribed boundary conditions of a given problem; however, among these there is only one physically possible profile which, by satisfying the appropriate Euler equation, makes the functional of the corresponding variational problem stationary. It is often difficult to guess which profile this is. Instead, we try approximate profiles for which the functional has a value approximately equal to its stationary value. As we learned in Chapter 2, in the selection of approximate profiles there are two procedures, the so-called *Ritz* and *Kantorovich* methods. We shall now examine in detail the Ritz method. The Kantorovich method, which is suitable for use with multi-dimensional and unsteady problems, will be discussed in Sections 8–8 through 8–11.

8–4. Ritz Method

This method is based on the selection of a convergent sequence of functions

$$y(x) = \sum_{n=0}^{N} a_n\varphi_n(x), \tag{8–33}$$

depending on N parameters, where $\varphi_n(x)$ for all values of n satisfies the boundary conditions; otherwise the choice of the functions $\varphi_n(x)$ is to a large extent arbitrary. In physical problems the general behavior of the desired solution is usually known and, depending on the nature of the problem, special functions such as polynomials or circular, hyperbolic, cylindrical, or spherical functions may conveniently be employed.

Inserting Eq. (8–33) into Eq. (8–2), we obtain the functional in terms of N parameters, $I(a_0, a_1, a_2, \ldots, a_N)$. The stationary values of this functional

may be obtained in the usual manner. The result is N simultaneous algebraic equations

$$\partial I/\partial a_n = 0, \qquad n = 0, 1, 2, \ldots, N, \tag{8–34}$$

sufficient for the evaluation of the parameters of Eq. (8–33). As may be expected, the approximate solution given by Eq. (8–33) becomes closer to the exact solution when the number N of the approximations is increased; the convergence of this solution largely depends on the choice of functions $\varphi_n(x)$.

A more elaborate procedure consists in a sequence of approximations

$$\begin{aligned} y_0(x) &= a_0\varphi_0(x), \\ y_1(x) &= a_0\varphi_0(x) + a_1\varphi_1(x), \\ &\vdots \\ y_N(x) &= a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_N\varphi_N(x), \end{aligned}$$

to be evaluated separately. With this procedure, we can check by successive comparison the accuracy attained at each order of approximation. Note that for the same coefficient different values are obtained, depending on the order of approximation.

Example 8–2. Let us obtain an approximate solution to the problem of Example (8–1) by means of the Ritz method.

In terms of polynomials, for example, the series

$$\theta(\xi)/\theta_0 = 1 - (1 - \xi^2)(a_0 + a_1\xi^2 + a_2\xi^4 + \cdots) \tag{8–35}$$

or

$$\theta(\xi)/\theta_0 = 1 - (1 - \xi^2)a_0 - (1 - \xi^2)^2a_1 - (1 - \xi^2)^3a_2 - \cdots, \tag{8–36}$$

both of which satisfy the boundary conditions of the problem, may be used conveniently. Here $\xi = x/L$ is the dimensionless distance and θ_0 the base temperature. The first of these series has already been used in Example 3–10 for an approximate solution to the same problem by the integral method; since this series involves less algebra than that of Eq. (8–36) in the evaluation of the unknown parameters a_n, it will also be used here. Thus we shall be able to compare the results of the integral method with those obtained by the variational technique.

Here we shall limit ourselves to the first- and second-order approximations, but we shall carry out both of them so that we may compare the accuracy attained at each order of approximation. Although Eq. (8–26) is the only variational form of the problem, Eq. (8–23), being identical to Eq. (8–26) and involving less algebra in the evaluation of the unknown parameters, is more convenient to use than Eq. (8–26).

The first approximation of Eq. (8–35),

$$\theta(\xi)/\theta_0 = 1 - (1 - \xi^2)a_0, \tag{3–195}$$

introduced into the dimensionless form of Eq. (8–26),

$$\int_0^1 \left(\frac{d^2\theta}{d\xi^2} - \mu^2\theta\right) \delta\theta \, d\xi = 0, \tag{8–37}$$

yields

$$-\int_0^1 \{2a_0 - \mu^2[1 - (1 - \xi^2)a_0]\}(1 - \xi^2)\, \delta a_0 \, d\xi = 0.$$

This integral may readily be evaluated by the successive use of Eq. (8–129).* The result is

$$[2(1 + 2\mu^2/5)a_0 - \mu^2]\, \delta a_0 = 0.$$

Since δa_0 is arbitrary, the coefficient in brackets must vanish, giving

$$a_0 = \frac{\mu^2/2}{1 + 2\mu^2/5}.$$

Thus a first-order approximation of the temperature distribution is

$$\frac{\theta(\xi)}{\theta_0} = 1 - \frac{\mu^2/2}{1 + 2\mu^2/5}(1 - \xi^2). \tag{8–38}$$

The second approximation of Eq. (8–35),

$$\theta(\xi)/\theta_0 = 1 - (1 - \xi^2)(a_0 + a_1\xi^2), \tag{3–215}$$

introduced into Eq. (8–37) yields

$$-\int_0^1 \{2a_0 - 2a_1 + 12a_1\xi^2 - \mu^2[1 - (1 - \xi^2)(a_0 + a_1\xi^2)]\} \times (1 - \xi^2)(\delta a_0 + \xi^2\, \delta a_1)\, d\xi = 0.$$

This integral may be evaluated by the repeated use of Eqs. (8–129) and (8–131). The result is

$$\tfrac{4}{3}[(1 + \tfrac{2}{5}\mu^2)a_0 + \tfrac{1}{5}(1 + \tfrac{2}{7}\mu^2)a_1 - \tfrac{1}{2}\mu^2]\, \delta a_0 + \tfrac{4}{15}[(1 + \tfrac{2}{7}\mu^2)a_0 + \tfrac{1}{7}(11 + \tfrac{2}{3}\mu^2)a_1 - \tfrac{1}{2}\mu^2]\, \delta a_1 = 0.$$

Since δa_0 and δa_1 are arbitrary, their coefficients must vanish, giving the simultaneous algebraic equations

$$(1 + \tfrac{2}{5}\mu^2)a_0 + \tfrac{1}{5}(1 + \tfrac{2}{7}\mu^2)a_1 = \tfrac{1}{2}\mu^2,$$
$$(1 + \tfrac{2}{7}\mu^2)a_0 + \tfrac{1}{7}(11 + \tfrac{2}{3}\mu^2)a_1 = \tfrac{1}{2}\mu^2.$$

---

* See Section 8–12.

Solving these equations for a_0 and a_1 yields

$$a_0 = \frac{\mu^2(3 + \mu^2/12)}{6 + 8\mu^2/3 + 2\mu^4/21}, \qquad a_1 = \frac{\mu^4/4}{6 + 8\mu^2/3 + 2\mu^4/21}.$$

Thus a second-order approximation of the temperature distribution is

$$\frac{\theta(\xi)}{\theta_0} = 1 - \frac{\mu^2(1 - \xi^2)}{6 + 8\mu^2/3 + 2\mu^4/21}\left[\left(3 + \frac{\mu^2}{12}\right) + \frac{\mu^2\xi^2}{4}\right]. \tag{8–39}$$

Let us now compare the first- and second-order approximations obtained by the variational calculus and the integral method with the exact solution of the problem. As the basis of comparison, we shall use the total heat loss from the fin, since this is more important and meaningful for the purpose than the tip temperature. Although the heat transfer by conduction from the base of the fin is equal to the heat loss by convection from the peripheral surface of the fin, an approximate profile results in different values for these. For obvious reasons, we choose to use the peripheral heat transfer, whose dimensionless form is

$$\frac{q}{kA\theta_0/L} = \mu^2 \int_0^1 \theta \, d\xi. \tag{8–40}$$

Inserting Eqs. (8–38) and (8–39) into Eq. (8–40) gives the first- and second-order approximations of the heat loss from the fin,

$$\frac{q}{kA\theta_0/L} = \frac{\mu^2(1 + \mu^2/15)}{(1 + 2\mu^2/5)}, \qquad \frac{q}{kA\theta_0/L} = \frac{\mu^2(3 + \mu^2/3 + \mu^4/315)}{(3 + 4\mu^2/3 + \mu^4/21)}. \tag{8–41}$$

In Table 8–1, we compare the exact heat loss from the fin with the results of the integral technique (see Table 3–4) and those of the variational calculus. This table clearly shows the improved accuracy of results obtained by use of the variational calculus as compared with those obtained by the integral method when the same profile is employed for both cases.

In the light of the foregoing problem, let us emphasize the basic difference between the variational calculus and the integral method. Suppose that we wish to find the nth approximation of a problem by either of the two methods. The variational calculus results directly in n simultaneous algebraic equations that yield the values of the n parameters, weighed to give the closest fit to the exact solution. On the other hand, as a result of integration of the governing differential equation, the integral method yields only one algebraic relation among the unknown parameters. The remaining $n - 1$ conditions are, to some extent, arbitrarily determined (recall Section 3–11). Therefore, the accuracy of the approximation obtained by the integral method depends on individual skill and experience. In general, this accuracy does not exceed that of the approximation obtained by the variational calculus.

TABLE 8–1

μ			$\frac{1}{2}$	1	2	4
Exact			0.2311	0.7616	1.9281	3.9973
Approximate	First	Integral Error, %	0.2308 0.130	0.7500 1.52	1.7143 11.1	2.5263 36.8
		Variational Error, %	0.2311 0.000	0.7619 0.041	1.9487 1.07	4.4685 11.8
	Second	Integral I Error, %	0.2311 0.000	0.7616 0.000	1.9269 0.062	3.9223 1.88
		Integral II Error, %	0.2311 0.000	0.7614 0.026	1.9130 0.783	3.6791 7.96
		Variational Error, %	0.2311 0.000	0.7616 0.000	1.9281 0.000	4.0066 0.233

Example 8–3. Reconsider Problem 3–30 (Fig. 8–5). Let us find the steady temperature distribution in the system. The case of constant wear readily accepts an exact solution; hence we focus our attention on the more involved case of constant pressure.

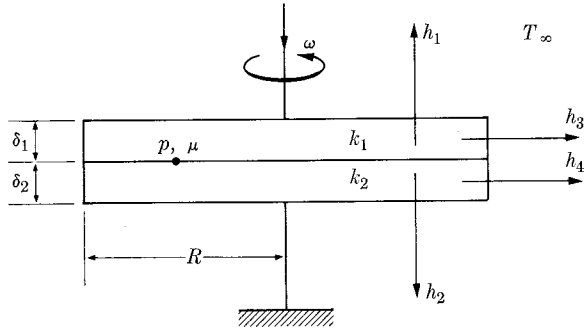


FIG. 8–5

Under steady conditions, with the axial temperature distribution neglected, the differential formulation of the problem is

$$\frac{d}{d\rho}\left(\rho \frac{d\theta}{d\rho}\right) - \mu^2 \rho\theta + \nu\rho^2 = 0, \tag{8–42}$$

where

$$\theta = T - T_\infty, \quad \rho = \frac{r}{R}, \quad \mu^2 = \left(\frac{h_1 + h_2}{k_1\,\delta_1 + k_2\,\delta_2}\right) R^2, \quad \nu = \frac{\mu p \omega R^3}{(k_1\,\delta_1 + k_2\,\delta_2)}.$$

The variational form of Eq. (8–42) is identical to

$$\int_0^1 \left[\frac{d}{d\rho}\left(\rho \frac{d\theta}{d\rho}\right) - \mu^2 \rho\theta + \nu\rho^2\right] \delta\theta \, d\rho = 0. \tag{8–43}$$

Let us neglect the heat transfer from the peripheral surface of the disks. Then a first approximation of the temperature distribution which satisfies the end conditions may be written in the form

$$\theta(\rho) = a_0(1 + 3\rho^2 - 2\rho^3). \tag{8–44}$$

Introducing Eq. (8–44) into Eq. (8–43), then integrating and rearranging the result gives

$$a_0 = \frac{(21/52)\nu}{21/52 + \mu^2}.$$

Thus a first-order approximation of the temperature distribution is

$$\frac{\theta(\rho)}{\nu} = \frac{21/52}{21/52 + \mu^2}(1 + 3\rho^2 - 2\rho^3). \tag{8–45}$$

Similarly, a second-order approximation of the temperature may be constructed in the form

$$\theta(\rho) = a_0 + (3\rho^2 - 2\rho^3)a_1. \tag{8–46}$$

Insertion of Eq. (8–46) into Eq. (8–43) and subsequent integration yields the simultaneous algebraic equations

$$\mu^2 a_0 + \tfrac{7}{10}\mu^2 a_1 = \tfrac{2}{3}\nu,$$
$$\tfrac{7}{20}\mu^2 a_0 + \tfrac{2}{7}(\mu^2 + \tfrac{21}{10})a_1 = \tfrac{4}{15}\nu.$$

Solving these equations for a_0 and a_1 gives

$$a_0 = \frac{16\nu(\mu^2 + 105)}{171\mu^2(\mu^2 + 840/57)}, \qquad a_1 = \frac{140\nu}{171(\mu^2 + 840/57)}.$$

Thus a second-order approximation of the temperature distribution is

$$\frac{\theta(\rho)}{\nu} = \frac{16(\mu^2 + 105)}{171\mu^2(\mu^2 + 840/57)} + \frac{140}{171(\mu^2 + 840/57)}(3\rho^2 - 2\rho^3). \tag{8–47}$$

Let us compare, for example, the centerline temperature of the first and second approximations to see how well the two approximations agree. Introducing $\rho = 0$ into Eqs. (8–45) and (8–47), and establishing the ratio of centerline temperatures yields

$$\frac{\theta_1(0)}{\theta_2(0)} = \frac{3591\mu^2(\mu^2 + 840/57)}{832(\mu^2 + 105)(\mu^2 + 21/52)}.$$

This ratio is given in Table 8–2 as a function of μ. Inspection of the table reveals a large and variable discrepancy between the two approximations which is quite contrary to the results of Example 8–2. For problems with no specified temperature at the boundaries, a first approximation such as Eq. (8–44), intro-

TABLE 8–2

μ	$\frac{1}{2}$	1	2	4
$\theta_1(0)/\theta_2(0)$	0.2350	0.4564	0.6739	1.069

ducing a constraint between the centerline and the peripheral temperatures, gives a rather poor result and should not be used. This point will be discussed at greater length in the next section. To illustrate the accuracy of the approximate solutions of the foregoing problem, the reader should carry out the third approximation and compare it with the second approximation given by Eq. (8–47) (see Problem 8–4).

8–5. Steady One-Dimensional Ritz Profiles

As may be expected, profiles subject to homogeneous boundary conditions are, in general, easier to construct than profiles subject to nonhomogeneous boundary conditions. Thus, using the principle of superposition, we first reduce the problem of selecting the latter type to that of selecting the former type. The procedure is best explained in terms of frequently encountered boundary conditions as follows:

(a) *The function is specified at the boundaries,*

$$y(0) = Y_2, \qquad y(L) = Y_1.$$

Let us separate the desired profile into three profiles as shown in Fig. 8–6. Thus we have

$$y(x) = y_1(x) + y_2(x) + y_3(x).$$

Each of the first two of these profiles satisfies one nonhomogeneous boundary condition of the problem. The selection of $y_1(x)$ and $y_2(x)$ as straight lines is for convenience only. It follows then that

$$y(x) = \left(\frac{x}{L}\right) Y_1 + \left(\frac{L - x}{L}\right) Y_2 + y_3(x), \tag{8-48}$$

where the last profile satisfies the corresponding homogeneous boundary conditions,

$$y_3(0) = 0, \qquad y_3(L) = 0.$$

The use of polynomials or circular functions in the construction of $y_3(x)$ readily gives

$$y_3(x) = x(L - x)(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{8-49}$$

or

$$y_3(x) = a_0 \sin(\pi x/L) + a_1 \sin(2\pi x/L) + a_2 \sin(3\pi x/L) + \cdots \tag{8-50}$$

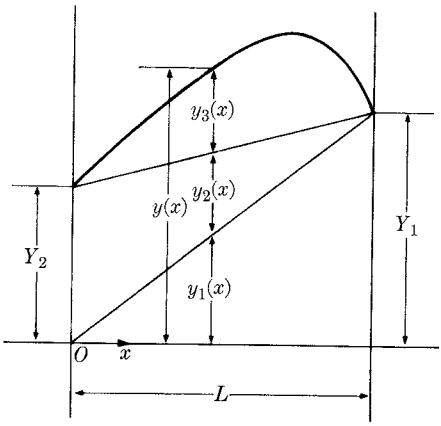


FIG. 8–6

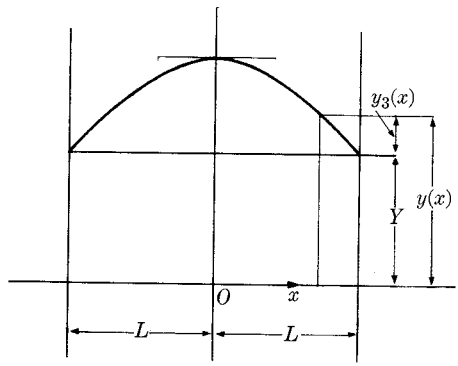


FIG. 8–7

As a special case, the same profiles may be geometrically and thermally symmetric (Fig. 8–7). In such a case, given the fact that $Y_1 = Y_2 = Y$, it follows that $y_1(x) = y_2(x) = Y$. Thus

$$y(x) = Y + y_3(x), \tag{8–51}$$

where

$$y_3(x) = (L^2 - x^2)(a_0 + a_1x^2 + a_2x^4 + \cdots) \tag{8–52}$$

or

$$y_3(x) = a_0 \cos(\pi x/2L) + a_1 \cos(3\pi x/2L) + a_2 \cos(5\pi x/2L) + \cdots \tag{8–53}$$

(b) *The gradient of the function is specified at the boundaries,*

$$y'(0) = m_2, \qquad y'(L) = m_1.$$

Again let the required profile be composed of three profiles,

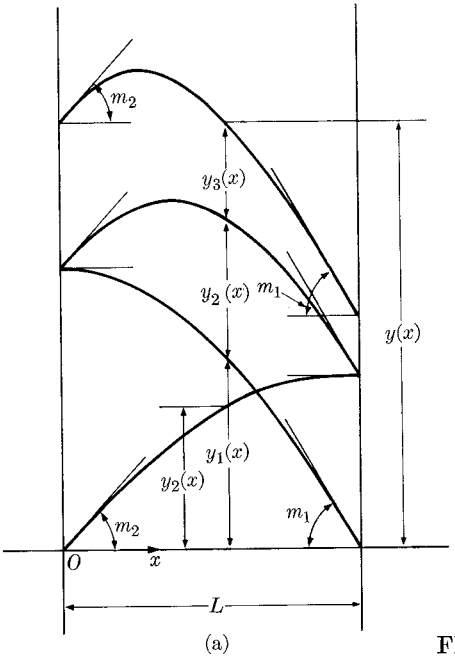
$$y(x) = y_1(x) + y_2(x) + y_3(x),$$

such that each of the first two satisfies one nonhomogeneous boundary condition and the homogeneous form of the other boundary condition, as shown in Fig. 8–8(a). The simplest physically possible profiles which satisfy these boundary conditions are parabolas. Thus we have

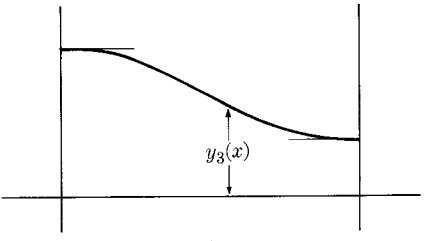
$$y(x) = \frac{x^2}{2L} m_1 - \frac{(L^2 - x^2)}{2L} m_2 + y_3(x), \tag{8–54}$$

where the third profile, $y_3(x)$, satisfies the corresponding homogeneous boundary conditions

$$y_3'(0) = 0, \qquad y_3'(L) = 0.$$



(a)



(b)

FIG. 8–8

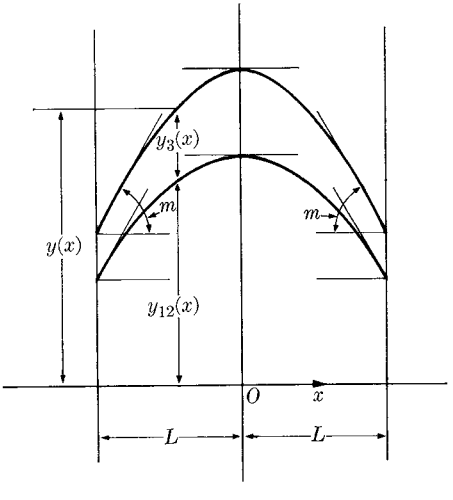
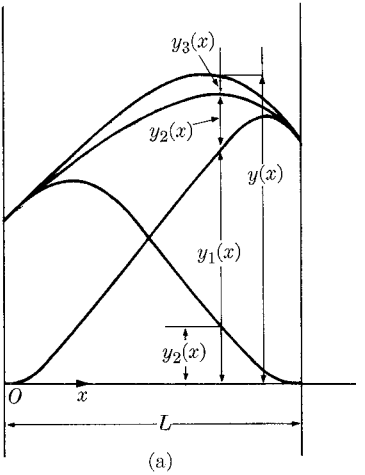
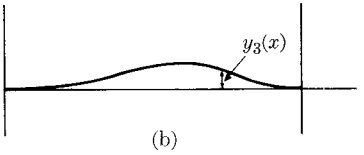


FIG. 8–9



(a)



(b)

FIG. 8–10

It follows from Fig. 8–8(b) that in terms of polynomials

$$y_3(x) = a_0 + a_1(3Lx^2 - 2x^3) + a_2(L - x)^2x^2 + a_3(L - x)^2x^3 + a_4(L - x)^2x^4 + \cdots \tag{8–55}$$

or

$$y_3(x) = a_0 + a_1(3Lx^2 - 2x^3) + a_2(4Lx^3 - 3x^4) + a_3(5Lx^4 - 4x^5) + \cdots \tag{8–56}$$

The use of circular functions results in

$$y_3(x) = a_0 + a_1 \cos(\pi x/L) + a_2 \cos(2\pi x/L) + a_3 \cos(3\pi x/L) + \cdots \tag{8–57}$$

The geometrically and thermally symmetric case of the same profiles is shown in Fig. 8–9. Since in this case $m_1 = m_2 = m$, the profiles $y_1(x)$ and $y_2(x)$ of the general case may now be replaced by a single profile, say $y_{12}(x)$. Thus

$$y(x) = y_{12}(x) + y_3(x).$$

Assuming that $y_{12}(x)$ is subject to the nonhomogeneous boundary condition of the problem, and satisfying this boundary condition again by a parabola, we have

$$y(x) = -\frac{(L^2 - x^2)}{2L} m + y_3(x), \tag{8–58}$$

where $y_3(x)$, having the general form of Fig. 8–8(b), may be represented by the profile of Eq. (8–56) or Eq. (8–57).

(c) *A linear combination of the function and its derivative is prescribed at the boundaries,*

$$y'(0) + B_2y(0) = n_2, \qquad y'(L) + B_1y(L) = n_1.$$

As before, we let

$$y(x) = y_1(x) + y_2(x) + y_3(x),$$

where $y_1(x)$ satisfies the second boundary condition and the homogeneous boundary condition corresponding to the first boundary condition, and $y_2(x)$ satisfies the first boundary condition and the homogeneous boundary condition corresponding to the second boundary condition (Fig. 8–10a). Note that the homogeneous boundary conditions are conveniently satisfied by letting the function and its derivative, rather than their sum, be separately equal to zero. Since $y_1(x)$ and $y_2(x)$ now have saddle points, the simplest physically meaningful profiles corresponding to these functions must necessarily be constructed in terms of third-order polynomials. Thus we have

$$y_1(x) = x^2(x + C_1) \qquad \text{and} \qquad y_2(x) = (L - x)^2(x + C_2),$$

where C_1 and C_2 are to be evaluated from the second and the first boundary conditions, respectively. The result is

$$y_1(x) = x^2\left(x - L - \frac{L - n_1/L}{2 + B_1 L}\right) \tag{8–59}$$

and

$$y_2(x) = (L - x)^2\left(x + \frac{L - n_2/L}{2 - B_2 L}\right). \tag{8–60}$$

The general form of $y_3(x)$ is shown in Fig. 8–10(b). It readily follows, if we use polynomials, that

$$y_3(x) = a_0(L - x)^2x^2 + a_1(L - x)^2x^3 + a_2(L - x)^2x^4 + \cdots, \tag{8–61}$$

or if we use circular functions, that

$$y_3(x) = a_0(1 - \cos \pi x/L) + a_1(1 - \cos 2\pi x/L) + a_2(1 - \cos 3\pi x/L) + \cdots \tag{8–62}$$

The geometrically and thermally symmetric case of these profiles is shown in Fig. 8–11(a). As in the preceding symmetric cases, we may write

$$y(x) = y_{12}(x) + y_3(x),$$

where $y_{12}(x)$ may be constructed in terms of the parabola

$$y_{12}(x) = C - x^2,$$

to be subjected to the boundary condition

$$y'(L) + By(L) = n.$$

Thus we have

$$y_{12}(x) = L^2 - x^2 + \frac{n + 2L}{B}. \tag{8–63}$$

The general form of $y_3(x)$ is shown in Fig. 8–11(b). Using polynomials, we obtain

$$y_3(x) = a_0 + a_1(L - x)^2x^2 + a_2(L - x)^2x^3 + a_3(L - x)^2x^4 + \cdots, \tag{8–64}$$

or using circular functions, we get

$$y_3(x) = a_0(1 + \cos \pi x/L) + a_1(1 + \cos 3\pi x/L) + a_2(1 + \cos 5\pi x/L) + \cdots \tag{8–65}$$

Having thus completed our investigation of steady one-dimensional problems we now proceed to the variational formulation of steady two-dimensional problems.

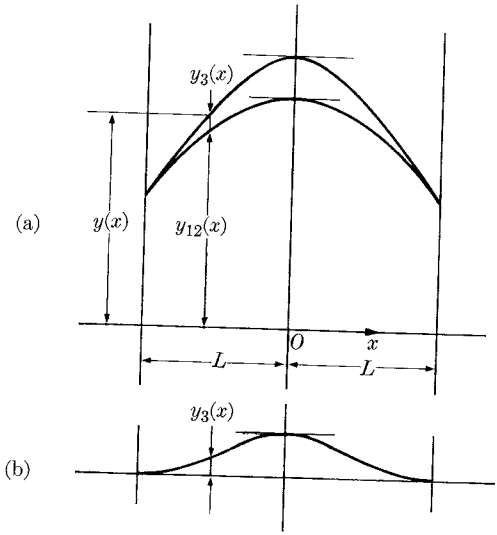


FIG. 8–11

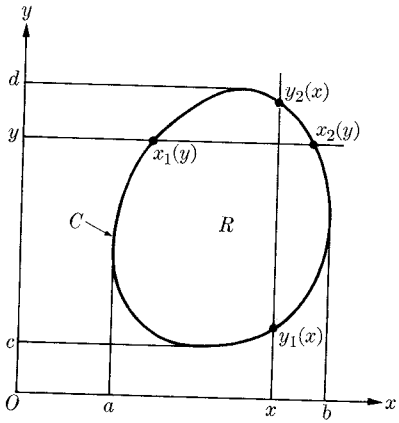


FIG. 8–12

8–6. Steady Two-Dimensional Problems

We begin our discussion by considering a two-dimensional (cartesian) functional

$$I = \iint_R F(x, y, T, T_x, T_y)\, dx\, dy, \tag{8–66}$$

where x and y are independent variables, $T(x, y)$ is the temperature function to be determined, and T_x, T_y are the partial derivatives with respect to x and y; the functional F and the integration domain R are known from the statement of the problem.

It follows from the rules of the variational calculus (Section 8–2) that the necessary condition, $\delta I = 0$, for the functional given by Eq. (8–66) to be stationary is

$$\delta I = \iint_R \left(\frac{\partial F}{\partial T}\, \delta T + \frac{\partial F}{\partial T_x}\, \delta T_x + \frac{\partial F}{\partial T_y}\, \delta T_y \right) dx\, dy = 0. \tag{8–67}$$

The Euler equation of this functional will now be obtained by expressing the last two terms of the integrand in terms of the variation δT. First we consider, for example, the last term of the integrand, which can be rearranged by using Fig. 8–12 as follows:

$$\iint_R \frac{\partial F}{\partial T_y}\, \delta T_y\, dx\, dy = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} \frac{\partial F}{\partial T_y}\, \delta T_y\, dy \right) dx. \tag{8–68}$$

Since

$$\delta T_y = \frac{\partial}{\partial y}(\delta T),$$

Eq. (8–68) integrated by parts in the y-direction gives

$$\iint_R \frac{\partial F}{\partial T_y}\,\delta T_y\,dx\,dy = \int_a^b \frac{\partial F}{\partial T_y}\,\delta T\Big|_{y_1(x)}^{y_2(x)} dx - \iint_R \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right)\delta T\,dx\,dy. \tag{8–69}$$

Similarly, the second term in the integrand of Eq. (8–67) results in

$$\iint_R \frac{\partial F}{\partial T_x}\,\delta T_x\,dx\,dy = \int_c^d \frac{\partial F}{\partial T_x}\,\delta T\Big|_{x_1(y)}^{x_2(y)} dy - \iint_R \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right)\delta T\,dx\,dy. \tag{8–70}$$

Using Fig. 8–13, we see that the first right-hand term of Eqs. (8–69) and (8–70) becomes

$$\int_a^b \frac{\partial F}{\partial T_y}\,\delta T\Big|_{y_1(x)}^{y_2(x)} dx + \int_c^d \frac{\partial F}{\partial T_x}\,\delta T\Big|_{x_1(y)}^{x_2(y)} dy = \oint_C \left(\frac{\partial F}{\partial T_x}\cos\phi + \frac{\partial F}{\partial T_y}\sin\phi\right)\delta T\,ds. \tag{8–71}$$

Combining Eqs. (8–67), (8–69), (8–70), and (8–71) yields the variational formulation

$$\delta I = \iint_R \left[\frac{\partial F}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right)\right]\delta T\,dx\,dy + \oint_C \left(\frac{\partial F}{\partial T_x}\cos\phi + \frac{\partial F}{\partial T_y}\sin\phi\right)\delta T\,ds = 0. \tag{8–72}$$

The natural boundary conditions of the problem, according to Eq. (8–72), satisfy

$$\oint_C \left(\frac{\partial F}{\partial T_x}\cos\phi + \frac{\partial F}{\partial T_y}\sin\phi\right)\delta T\,ds = 0. \tag{8–73}$$

One way of satisfying Eq. (8–73) is to select T such that it is continuously differentiable in R, and δT vanishes on C. This may be accomplished, for example, by specifying T on C. The variational formulation, subject to the natural boundary conditions, then becomes

$$\delta I = \iint_R \left[\frac{\partial F}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right)\right]\delta T\,dx\,dy = 0, \tag{8–74}$$

which implies that

$$\frac{\partial F}{\partial T} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial T_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial T_y}\right) = 0. \tag{8–75}$$

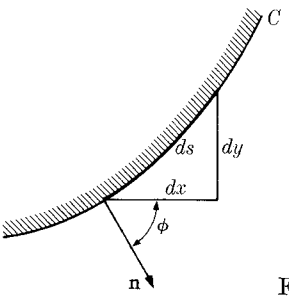


FIG. 8–13

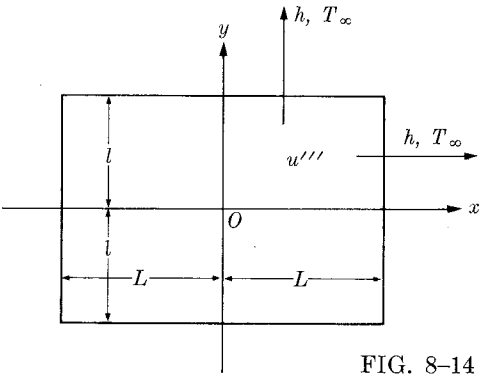


FIG. 8–14

Equation (8–75) is the Euler equation associated with the functional given by Eq. (8–66). Thus, in a manner analogous to the one-dimensional case, $T(x, y)$, subject to the natural boundary conditions, makes the integral of Eq. (8–66) stationary, if it satisfies the corresponding Euler equation.

Example 8–4. Reconsider Example 2–1 (Fig. 8–14). Assume the case of a large heat transfer coefficient. Let us find the steady variational formulation of the problem.

The governing differential equation is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{u'''}{k} = 0,$$

subject to the boundary conditions

$$\theta(\pm L, y) = \theta(x, \pm l) = 0,$$

where θ is measured above the ambient temperature T_∞.

The general procedure outlined in Eqs. (8–66) through (8–75) is now followed in the reverse order; taking the differential equation of the problem as the Euler equation of the variational formulation, we have from Eq. (8–74)

$$\delta I = \int_{-L}^{L}\int_{-l}^{l}\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{u'''}{k}\right)\delta\theta\, dx\, dy = 0. \tag{8–76}$$

The first term in the integrand of Eq. (8–76) integrated by parts in the x-direction results in

$$\int_{-L}^{L}\int_{-l}^{l}\frac{\partial^2\theta}{\partial x^2}\,\delta\theta\, dx\, dy = \int_{-l}^{l}\left(\int_{-L}^{L}\frac{\partial^2\theta}{\partial x^2}\,\delta\theta\, dx\right)dy$$

$$= \int_{-l}^{l}\frac{\partial\theta}{\partial x}\,\delta\theta\Big|_{-L}^{L} dy - \int_{-l}^{l}\int_{-L}^{L}\delta\left[\frac{1}{2}\left(\frac{\partial\theta}{\partial x}\right)^2\right]dx\, dy. \tag{8–77}$$

Similarly, the second term integrated by parts in the y-direction yields

$$\int_{-L}^{L}\int_{-l}^{l}\frac{\partial^2\theta}{\partial y^2}\,\delta\theta\,dx\,dy = \int_{-L}^{L}\frac{\partial\theta}{\partial y}\,\delta\theta\Big|_{-l}^{l}dx - \int_{-L}^{L}\int_{-l}^{l}\delta\left[\frac{1}{2}\left(\frac{\partial\theta}{\partial y}\right)^2\right]dx\,dy. \tag{8–78}$$

Furthermore, note that

$$\int_{-l}^{l}\frac{\partial\theta}{\partial x}\,\delta\theta\Big|_{-L}^{L}dy + \int_{-L}^{L}\frac{\partial\theta}{\partial y}\,\delta\theta\Big|_{-l}^{l}dx = \oint_C\frac{\partial\theta}{\partial n}\,\delta\theta\,ds, \tag{8–79}$$

where n is the outward normal to the periphery of the rectangle, ds the differential arc-length on this periphery, and C the boundaries of the rectangle (see Fig. 8–13). Finally, combining Eqs. (8–77), (8–78), (8–79), and introducing the result into Eq. (8–76) gives the variational form

$$\delta\iint_R\left\{-\frac{1}{2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right]+\frac{u'''}{k}\,\theta\right\}dx\,dy + \int_C\frac{\partial\theta}{\partial n}\,\delta\theta\,ds = 0. \tag{8–80}$$

Inspection of the line integral of Eq. (8–80) reveals that, as with steady one-dimensional problems, either a prescribed surface temperature or an insulated boundary will make this integral vanish; hence both are natural boundary conditions by definition. Since the temperature of the boundaries is prescribed in the present case, the variational form of the problem reduces to

$$\delta\iint_R\left\{-\frac{1}{2}\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2\right]+\frac{u'''}{k}\,\theta\right\}dx\,dy = 0. \tag{8–81}$$

8–7. Steady Two-Dimensional Ritz Method

The Ritz method of steady one-dimensional problems may readily be extended to steady two-dimensional problems. We select an approximate solution $\theta(x, y)$ which depends on N parameters and has the form of a convergent sequence of functions

$$\theta(x, y) = \sum_{n=0}^{N} a_n\varphi_n(x, y). \tag{8–82}$$

Here $\varphi_n(x, y)$ for all values of n satisfies the boundary conditions. Furthermore, if $\varphi_n(x, y)$ is assumed to be a product consisting of a function of x alone and a function of y alone,

$$\varphi_n(x, y) = X_n(x)Y_n(y),$$

then $X_n(x)$ and $Y_n(y)$ may be constructed such that $X_n(x)$ satisfies only the

boundary conditions in the x-direction and $Y_n(y)$ only the boundary conditions in the y-direction. Thus Eq. (8–82) becomes

$$\theta(x, y) = \sum_{n=0}^{N} a_n X_n(x) Y_n(y). \tag{8–83}$$

Example 8–5. Using the Ritz method, let us find an approximate solution of Example 8–4.

In terms of polynomials, for example, the series

$$\theta(x, y) = (L^2 - x^2)(l^2 - y^2)(a_0 + a_1 x^2 + a_2 y^2 + \cdots) \tag{8–84}$$

or

$$\theta(x, y) = a_0(L^2 - x^2)(l^2 - y^2) + a_1(L^2 - x^2)^2(l^2 - y^2)^2 + \cdots, \tag{8–85}$$

both of which satisfy the boundary conditions of the problem, may be used conveniently. Because of the thermal and geometric symmetry of the problem with respect to the selected reference frame, odd powers of x and y, being identically zero, are excluded from the series of Eqs. (8–84) and (8–85). The first of these series has already been used in Examples 2–1 and 4–11 in connection with the integral formulation and its solution. The same series, permitting a comparison between the results of the integral and variational techniques, will also be used here.

Again, the first two approximations are carried out. The first approximation of Eqs. (8–84),

$$\theta(x, y) = (L^2 - x^2)(l^2 - y^2)a_0, \tag{2–134}$$

inserted into Eq. (8–76) yields

$$4\int_0^L \int_0^l [-2(l^2 - y^2)a_0 - 2(L^2 - x^2)a_0 + u'''/k] \times (L^2 - x^2)(l^2 - y^2)\, \delta a_0\, dx\, dy = 0.$$

The result of this integral gives

$$a_0 = \frac{5}{8} \frac{u'''/k}{(l^2 + L^2)}.$$

Thus the first-order Ritz profile based on the variational formulation is

$$\frac{\theta(x, y)}{u'''l^2/k} = \frac{5}{8} \frac{[1 - (x/L)^2][1 - (y/l)^2]}{1 + (l/L)^2}. \tag{8–86}$$

Similarly, the second approximation of Eq. (8–84),

$$\theta(x, y) = (L^2 - x^2)(l^2 - y^2)(a_0 + a_1 x^2), \tag{4–135}$$

introduced into Eq. (8–76) gives

$$4\int_0^L \int_0^l [-2(l^2 - y^2)(a_0 - a_1L^2 + 6a_1x^2) - 2(L^2 - x^2)(a_0 + a_1x^2) + u'''/k] \times (L^2 - x^2)(l^2 - y^2)(\delta a_0 + x^2\,\delta a_1)\,dx\,dy = 0.$$

This integral yields the simultaneous algebraic equations

$$(l^2 + L^2)a_0 + \left(\frac{l^2}{5} + \frac{L^2}{7}\right)L^2a_1 = \frac{5}{8}\frac{u'''}{k},$$

$$\left(\frac{l^2}{5} + \frac{L^2}{7}\right)a_0 + \frac{1}{7}\left(\frac{11}{5}l^2 + \frac{L^2}{3}\right)L^2a_1 = \frac{1}{8}\frac{u'''}{k}.$$

Solving these equations for a_0 and a_1 gives

$$a_0 = \frac{(u'''/16k)(L^2/3 + 24l^2/5)}{(L^4/21 + 8L^2l^2/15 + 12l^4/25)}, \qquad a_1 = \frac{(u'''/16k)}{(L^4/21 + 8L^2l^2/15 + 12l^4/25)}.$$

Hence the second-order Ritz profile based on the variational formulation is

$$\frac{\theta(x, y)}{u'''l^2/k} = \frac{1}{16}\left[1 - \left(\frac{x}{L}\right)^2\right]\left[1 - \left(\frac{y}{l}\right)^2\left[\frac{1/3 + (24/5)(l/L)^2 + (x/L)^2}{1/21 + (8/15)(l/L)^2 + (12/25)(l/L)^4}\right]\right]. \tag{8–87}$$

Before comparing the foregoing approximate solutions with the solution by the integral method and the exact solution, let us consider the Kantorovich method, to which we have so far given no attention.

8–8. Kantorovich Method

Up to now, we have constructed two-dimensional Ritz profiles in the form of Eq. (8–83) by specifying completely the functions $X_n(x)$ and $Y_n(y)$ in the x- and y-directions. However, in many physical problems, the general behavior of the unknown temperature $\theta(x, y)$ in one direction, say the y-direction, may be predicted more accurately than that in the other direction. For such cases, the Ritz profile may be generalized by specifying completely the functions $Y_n(y)$, but leaving the functions $X_n(x)$ unspecified, to be determined by the variational calculus. The use of this profile in variational formulations leads to a set of ordinary differential equations in terms of $X_n(x)$, to be considered with proper boundary conditions in the x-direction. This procedure constitutes the basis of the *Kantorovich method.*

Example 8–6. Using the Kantorovich method, let us find an approximate solution of Example 8–4.

If we approximate the temperature dependence in the y-direction by, for example, a parabola, the function

$$\theta(x, y) = (l^2 - y^2)X(x), \tag{2–137}$$

which satisfies the boundary conditions of the problem in the y-direction, may be used as a first approximation. Since a function over a short interval can be approximated closely by a specified function, when $L > l$ Eq. (2–137) is suitable. On the other hand, when $l > L$ the function

$$\theta(x, y) = (L^2 - x^2)Y(y)$$

should be used instead. Similarly, since a function over a long interval can be approximated more closely by higher-degree polynomials, when $L > l$ the function

$$\theta(x, y) = (L^2 - x^2)(l^2 - y^2)(a_0 + a_1x^2) \tag{4–135}$$

should be used for the second approximation of the Ritz profile; but when $l > L$ the function

$$\theta(x, y) = (L^2 - x^2)(l^2 - y^2)(a_0 + a_1y^2)$$

is more suitable. The preceding remarks point once again to the fact that the accuracy of an approximate procedure rests heavily on the individual's understanding of the physics of the problem and on his skill in the choice of profiles.

Returning now to the solution of the problem by the Kantorovich method, let us insert Eq. (2–137) into Eq. (8–76). This gives

$$4\int_0^L \int_0^l [(l^2 - y^2)X'' - 2X + u'''/k](l^2 - y^2)\,\delta X\,dx\,dy = 0.$$

Integrating this integral in the y-direction, we obtain

$$\left[\tfrac{8}{3}l^3 \int_0^L (\tfrac{4}{5}l^2X'' - 2X + u'''/k)\,dx\right]\delta X = 0. \tag{8–88}$$

Since δX is arbitrary, the integral of Eq. (8–88) must vanish, which implies that the integrand be identically zero. Thus the unspecified function $X(x)$ is found to satisfy the differential equation

$$X'' - \frac{5}{2l^2}X = -\frac{5u'''}{4kl^2}, \tag{8–89}$$

subject to the boundary conditions

$$X'(0) = 0, \qquad X(L) = 0. \tag{8–90}$$

The solution of Eq. (8–89) which satisfies Eq. (8–90) is

$$X(x) = \frac{u'''}{2k}\left(1 - \frac{\cosh \sqrt{\tfrac{5}{2}}(x/l)}{\cosh \sqrt{\tfrac{5}{2}}(L/l)}\right).$$

Thus a first-order Kantorovich profile obtained by the use of the variational form of the problem is

$$\frac{\theta(x, y)}{u'''l^2/k} = \frac{1}{2}\left[1 - \left(\frac{y}{l}\right)^2\right]\left(1 - \frac{\cosh \sqrt{\tfrac{5}{2}}(x/l)}{\cosh \sqrt{\tfrac{5}{2}}(L/l)}\right). \tag{8–91}$$

Similarly, the second-order Kantorovich profile from Example 4–11,

$$\theta(x, y) = (l^2 - y^2)[X_1(x) + y^2 X_2(x)], \tag{4–141}$$

introduced into Eq. (8–76) gives

$$4\int_0^L \int_0^l [(l^2 - y^2)(X_1'' + y^2 X_2'') - 2X_1 + 2X_2(l^2 - 6y^2) + u'''/k] \times (l^2 - y^2)(\delta X_1 + y^2\, \delta X_2)\, dx\, dy = 0.$$

Integrating this integral in the y-direction, we obtain

$$\tfrac{8}{3}l^3\left[\int_0^L \left(\tfrac{4}{5}l^2 X_1'' + \tfrac{4}{35}l^4 X_2'' - 2X_1 - \tfrac{2}{5}l^2 X_2 + \frac{u'''}{k}\right) dx\right]\delta X_1 + \tfrac{8}{15}l^5\left[\int_0^L \left(\tfrac{4}{7}l^2 X_1'' + \tfrac{4}{21}l^4 X_2'' - 2X_1 - \tfrac{22}{7}l^2 X_2 + \frac{u'''}{k}\right) dx\right]\delta X_2 = 0. \tag{8–92}$$

Since δX_1 and δX_2 are arbitrary, the integrals of Eq. (8–92) must vanish, which implies that the integrands be identically zero. Then $X_1(x)$ and $X_2(x)$ satisfy the simultaneous differential equations

$$\begin{aligned} \tfrac{4}{5}l^2 X_1'' + \tfrac{4}{35}l^4 X_2'' - 2X_1 - \tfrac{2}{5}l^2 X_2 + \frac{u'''}{k} &= 0, \\ \tfrac{4}{7}l^2 X_1'' + \tfrac{4}{21}l^4 X_2'' - 2X_1 - \tfrac{22}{7}l^2 X_2 + \frac{u'''}{k} &= 0. \end{aligned} \tag{8–93}$$

Again, by introduction of the linear operator $D \equiv d/dx$ Eq. (8–93) may be written in the equivalent operational form

$$\begin{aligned} 2(\tfrac{2}{5}l^2 D^2 - 1)X_1 + \tfrac{2}{5}l^2(\tfrac{2}{7}l^2 D^2 - 1)X_2 &= -\frac{u'''}{k}, \\ 2(\tfrac{2}{7}l^2 D^2 - 1)X_1 + \tfrac{2}{7}l^2(\tfrac{2}{3}l^2 D^2 - 11)X_2 &= -\frac{u'''}{k}. \end{aligned}$$

Using the theory of determinants, or by direct elimination, we obtain the following differential equations for X_1 and X_2:

$$\begin{aligned}(l^4D^4 - 28l^2D^2 + 63)X_1 &= (63/2)u'''/k,\\ (l^4D^4 - 28l^2D^2 + 63)X_2 &= 0.\end{aligned} \tag{8–94}$$

The homogeneous parts of X_1 and X_2 have the same characteristic equation

$$l^4r^4 - 28l^2r^2 + 63 = 0,$$

yielding four real roots, $\pm\alpha/l$, $\pm\beta/l$, where $\alpha = [14 - (133)^{1/2}]^{1/2}$ and $\beta = [14 + (133)^{1/2}]^{1/2}$. Thus four independent solutions, conveniently expressed in terms of hyperbolic functions, are $\cosh \alpha(x/l)$, $\sinh \alpha(x/l)$, $\cosh \beta(x/l)$, $\sinh \beta(x/l)$. The thermal and geometric symmetry of the problem with respect to the selected reference frame eliminates $\sinh \alpha(x/l)$ and $\sinh \beta(x/l)$ from the solution. Furthermore, noting the particular solutions $X_1 = u'''/2k$, $X_2 = 0$, we may write the general solutions of X_1 and X_2 in the form

$$\begin{aligned}X_1 &= u'''/2k + A \cosh \alpha(x/l) + B \cosh \beta(x/l),\\ X_2 &= C \cosh \alpha(x/l) + D \cosh \beta(x/l).\end{aligned} \tag{8–95}$$

To determine the relationship which must exist among A, B, C, and D, Eq. (8–95) is inserted into either one of the differential equations given by Eq. (8–93). Using the first equation, for example, and requiring that the result to be identically zero, we get

$$(\tfrac{2}{5}\alpha^2 - 1)A + \tfrac{1}{5}l^2(\tfrac{2}{7}\alpha^2 - 1)C = 0, \qquad (\tfrac{2}{5}\beta^2 - 1)B + \tfrac{1}{5}l^2(\tfrac{2}{7}\beta^2 - 1)D = 0.$$

Expressing, say C in terms of A, and D in terms of B, and introducing the result into the second equation of Eq. (8–95), we have

$$\begin{aligned}X_1 &= \frac{u'''}{2k} + A \cosh \alpha \frac{x}{l} + B \cosh \beta \frac{x}{l},\\ X_2 &= -\frac{5}{l^2}\left(\frac{2\alpha^2/5 - 1}{2\alpha^2/7 - 1}\right) A \cosh \alpha \frac{x}{l} - \frac{5}{l^2}\left(\frac{2\beta^2/5 - 1}{2\beta^2/7 - 1}\right) B \cosh \beta \frac{x}{l}.\end{aligned} \tag{8–96}$$

Since the temperature symmetry in the x-direction has already been used, the remaining condition, $\theta(L, y) = 0$, is now satisfied by requiring that

$$X_1(L) = 0, \qquad X_2(L) = 0.$$

Using this condition, from Eq. (8–96) we obtain

$$\begin{aligned}A \cosh \alpha \frac{L}{l} + B \cosh \beta \frac{L}{l} &= -\frac{u'''}{2k},\\ \left(\frac{2\alpha^2/5 - 1}{2\alpha^2/7 - 1}\right) A \cosh \alpha \frac{L}{l} + \left(\frac{2\beta^2/5 - 1}{2\beta^2/7 - 1}\right) B \cosh \beta \frac{L}{l} &= 0.\end{aligned}$$

Solving these equations for A and B results in

$$A = -\frac{35}{4}\left(\frac{q'''}{2k}\right)\frac{(2\beta^2/5 - 1)(2\alpha^2/7 - 1)}{(\alpha^2 - \beta^2)\cosh \alpha\,(L/l)},$$
$$B = \frac{35}{4}\left(\frac{q'''}{2k}\right)\frac{(2\alpha^2/5 - 1)(2\beta^2/7 - 1)}{(\alpha^2 - \beta^2)\cosh \beta\,(L/l)}. \tag{8–97}$$

Finally, combining Eqs. (4–141), (8–96), and (8–97), we find that a second-order Kantorovich profile based on the variational formulation of the problem is

$$\frac{\theta(x, y)}{u'''l^2/k} = \frac{1}{2}\left[1 - \left(\frac{y}{l}\right)^2\right]$$
$$\times \left\{1 + \frac{35}{4}\left(\frac{2\beta^2/5 - 1}{\beta^2 - \alpha^2}\right)[(2\alpha^2/7 - 1) - 5(2\alpha^2/5 - 1)(y/l)^2]\right.$$
$$\times \frac{\cosh \alpha(x/l)}{\cosh \alpha(L/l)} - \frac{35}{4}\left(\frac{2\alpha^2/5 - 1}{\beta^2 - \alpha^2}\right)$$
$$\left.\times [(2\beta^2/7 - 1) - 5(2\beta^2/5 - 1)(y/l)^2]\frac{\cosh \beta(x/l)}{\cosh \beta(L/l)}\right\}. \tag{8–98}$$

A comparison can now be made between the exact solution of the problem and the approximate Ritz and Kantorovich profiles obtained by the integral and variational techniques. To simplify the algebra of this comparison, only the first approximations will be taken into account, and the space dependence will be eliminated by considering a single location only. For this location let us take the origin of the reference frame, noting that the maximum error will occur at this point (why?). The exact and first-order approximate values of the origin temperature are obtained by inserting $x = 0$, $y = 0$ into Eqs. (4–133), (2–136), (2–143), (8–86), and (8–91). The comparison is summarized in Table 8–3 for three values of L/l.

Tables 8–1 and 8–3, and the experience gained in the integral and variational techniques from the previous sections, lead us to the following conclusions:

(a) In terms of a given profile, the accuracy attained by the variational calculus is better than that of the integral method.

(b) Kantorovich profiles employed with the variational or integral techniques give more accurate results than Ritz profiles, because only part of any Kantorovich profile is chosen *a priori*, the remainder being determined in accordance with the physics of the problem.

(c) It can be shown, using Example 8–6 and Table 8–3, for example, that the accuracy of a Ritz profile decreases with increasing L/l for two-dimensional problems. Clearly, if l is assumed constant but L increasing, the actual temperature distribution becomes difficult to approximate closely by a selected simple

profile. Thus the limiting one-dimensional problem which corresponds to $L/l = \infty$ cannot be obtained from the two-dimensional Ritz profile by simply letting $L/l \to \infty$ in this profile. By contrast, the Kantorovich profile converges rapidly to the actual distribution with increasing L/l, because the differential equation obtained by the Kantorovich technique accepts a particular solution as $L/l \to \infty$. Furthermore, if the y-direction of the two-dimensional Kantorovich profile is properly chosen, which was the case in Example 8–6, this profile approaches the exact one-dimensional solution as $L/l \to \infty$.

The reason the Kantorovich method yields a relatively poor result for $L/l = 1$, compared to the results obtained by the same method for $L/l > 1$, is that for $L/l = 1$ the actual temperature distribution is symmetric, hence cannot well be approximated by Kantorovich profiles, which are by nature asymmetric.

TABLE 8–3

L/l		1	2	4
Exact		0.29469	0.45687	0.49807
First Approximations	Integral-Ritz	0.37500	0.60000	0.70584
	Error, %	27.3	31.3	41.7
	Variational-Ritz	0.31250	0.50000	0.58824
	Error, %	6.05	9.44	18.1
	Integral-Kantorovich	0.32844	0.48723	0.49902
	Error, %	11.5	6.64	0.190
	Variational-Kantorovich	0.30259	0.45774	0.49821
	Error, %	2.68	0.189	0.0269

(d) Circular or other special functions may be used as well as polynomials in constructing approximate profiles. Although it is difficult to comment on the relative accuracy of these profiles, experience indicates that greater accuracy is obtained using polynomials as compared with circular functions (see Problems 8–1, 8–6). The reason for this is not quite clear. However, the following argument given in terms of Example 8–2 may be helpful. If only the first approximation is desired, polynomials may be more natural than circular functions, since the exact solution of a large number of simple conduction problems is found in terms of polynomials rather than of circular functions. Furthermore, because of the oscillating character of the higher circular harmonics (Fig. 8–15), the convergence of the circular functions is slower than that of polynomials. Nevertheless, circular profiles have two advantages: First, if

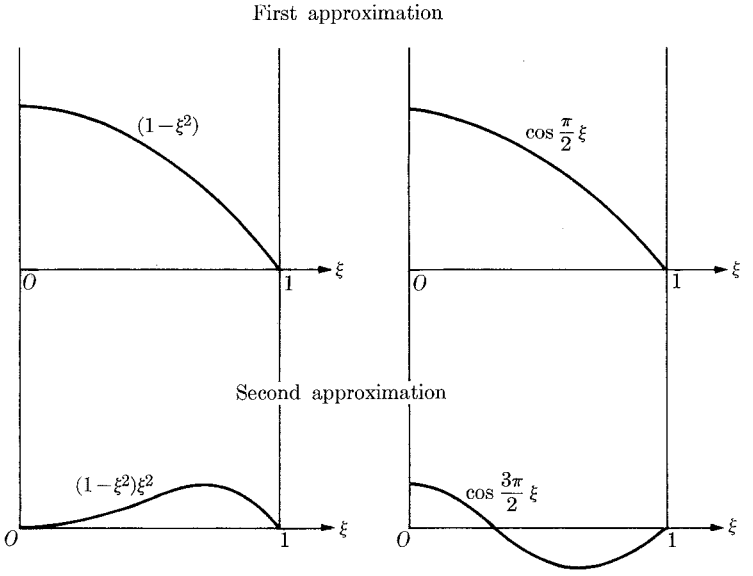


FIG. 8–15

these are selected such that they constitute a set of orthogonal functions, the evaluation of unknown parameters involves less algebra, because the products of different harmonics vanish; second, the values of unknown parameters do not depend on the order of approximation, since they are the Fourier coefficients in the expansion of the exact solution (see Problem 8–1).

8–9. Kantorovich Method Extended

The Kantorovich method is conveniently applied to a general class of steady two-dimensional problems characterized by the internal energy generation $u'''(x, y)$ in a region of the x-y-plane that is bounded by the curves $y = \pm f(x)$ and the straight lines $x = a$, $x = b$ on which the temperature remains constant, say zero (Fig. 8–16).

Using the basic concept of the Kantorovich method, we may construct a first approximation of the temperature distribution by leaving the x-direction to be determined by an unknown function $X(x)$ and satisfying the y-direction by the equations of the boundaries $y = \pm f(x)$. Thus we have

$$\theta(x, y) = (f^2 - y^2)X(x). \tag{8–99}$$

Introducing Eq. (8–99) into Eq. (8–76) gives

$$\int_a^b \int_{-f}^{f} [(f^2 - y^2)X'' + 4ff'X' + 2(f'^2 + ff'' - 1)X + u'''/k] \times (f^2 - y^2)\,\delta X\,dx\,dy = 0,$$

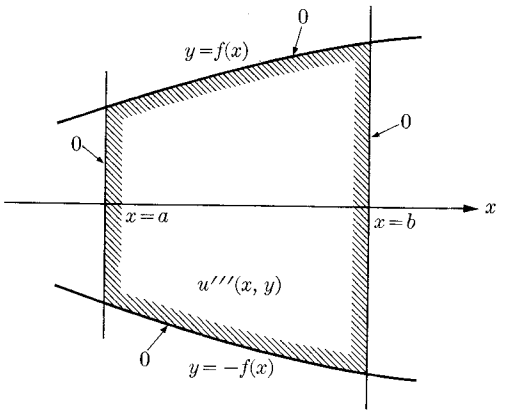


FIG. 8–16

and integrating this equation in the y-direction, we obtain the equation to be satisfied by X as follows:

$$\tfrac{2}{5}f^2X'' + 2ff'X' + (f'^2 + ff'' - 1)X = 3\Phi/8kf^3, \tag{8–100}$$

where

$$\Phi(x) = -\int_{-f}^{f} u'''(x, y)(f^2 - y^2)\,dy.$$

In particular, when $u'''(x, y)$ is constant Eq. (8–100) reduces to

$$\tfrac{2}{5}f^2X'' + 2ff'X' + (f'^2 + ff'' - 1)X = -u'''/2k. \tag{8–101}$$

Three special cases of Eq. (8–101) are considered below:

(a) $f(x) = l = \text{const}$ results in

$$X'' - \frac{5}{2l^2}X = -\frac{5u'''}{4kl^2}, \tag{8–89}$$

which is identical to Eq. (8–89) of Example 8–6, as expected.

(b) $f(x) = mx$ yields the Euler (Cauchy) equation in the form

$$x^2X'' + 5xX' + \frac{5(m^2 - 1)}{2m^2}X = -\frac{5u'''}{4km^2}, \tag{8–102}$$

whose solution is given in Section 3–3 (see Table 3–1).

(c) $f(x) = mx^n$, $n \neq 0$, $n \neq 1$ yields an equation reducible to a form of the Bessel equation. The procedure, being rather involved, is not given here.

From Chapter 2 until now we have solved a number of problems by the integral and variational techniques, as well as by the methods of exact solution. Thus we have been able to compare the accuracy of different approximate

methods. We have also become aware of the *convenience* of using approximate methods for problems with *complicated exact solutions,** and of the *importance* of using them *for those with no exact solutions.* We now show how approximate methods are easily applied to problems whose exact solution is not available.

Example 8–7. The uniform internal energy u''' is generated in a triangular bar (Fig. 8–17). The surfaces of the bar are kept at zero temperature. We wish to find the steady temperature of the bar.

According to the selected reference frame, $f(x) = x$ and $m = 1$. Equation (8–102) gives then

$$x^2X'' + 5xX' = -5u'''/4k. \quad (8\text{–}103)$$

The general solution of Eq. (8–103) is

$$X(x) = A + \frac{B}{x^4} - \frac{5u'''}{16k} \ln x. \quad (8\text{–}104)$$

Since

$$\lim_{x \to 0} x^2 \ln x \to 0,$$

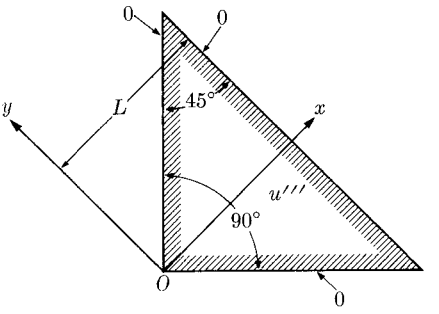


FIG. 8–17

the condition of finite temperature at the origin, $\theta(0, 0) \to$ finite, considered with the combination of Eqs. (8–99) and (8–104) yields $B = 0$. Then the other boundary condition, $X(L) = 0$, gives

$$A = (5u'''/16k) \ln L.$$

Introducing the values of A and B into Eq. (8–104) and the result into Eq. (8–99) yields the first-order Kantorovich profile

$$\frac{\theta(x, y)}{u'''L^2/k} = \frac{5}{16}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{y}{L}\right)^2\right] \ln\left(\frac{L}{x}\right). \quad (8\text{–}105)$$

The same problem may also be solved by the Ritz method. Thus we assume a profile in the form

$$\theta(x, y) = (x^2 - y^2)(L - x)a_0, \quad (8\text{–}106)$$

which satisfies all the boundaries; introducing this profile into Eq. (8–76) yields

$$\int_0^L \int_{-x}^{x} (-4a_0x + u'''/k)(x^2 - y^2)(L - x)\, \delta a_0\, dx\, dy = 0. \quad (8\text{–}107)$$

Integrating Eq. (8–107) with respect to x and y gives

$$a_0 = 6u'''/16kL.$$

* Recall the exact solution of Problem 3–30 for the case of constant pressure.

Introducing the value of a_0 into Eq. (8–106) results in the first-order Ritz profile

$$\frac{\theta(x, y)}{u'''L^2/k} = \frac{6}{16}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{y}{L}\right)^2\right]\left(1 - \frac{x}{L}\right). \tag{8–108}$$

Note that if we expand $\ln (L/x)$ into a Taylor series around $x/L = 1$ as

$$\ln (L/x) = -\ln (x/L) = (1 - x/L) + \tfrac{1}{2}(1 - x/L)^2 - \cdots,$$

then Eq. (8–105) may be rearranged in the form

$$\frac{\theta(x, y)}{u'''L^2/k} = \frac{5}{16}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{y}{L}\right)^2\right]\left[\left(1 - \frac{x}{L}\right) + \frac{1}{2}\left(1 - \frac{x}{L}\right)^2 - \cdots\right]. \tag{8–109}$$

Thus the Ritz profile given by Eq. (8–108) corresponds, except for a constant, to the first term in the power series expansion of the Kantorovich profile, Eq. (8–109). A similar relationship may be shown to exist between the Ritz and Kantorovich profiles of other problems.

8–10. Construction of Steady Two-Dimensional Profiles

Let us now consider the general rules useful in constructing steady two-dimensional profiles:

(a) *The Ritz profile of problems involving zero temperature on the boundaries.* This case may be reduced to that of finding the equation of the boundaries with respect to the selected reference frame. For example, a suitable Ritz profile for the steady two-dimensional problem given by Fig. 8–18 may readily be written in the form

$$\theta(x, y) = xy(1 - x/L - y/b)(1 + x/a - y/l)(a_0 + a_1x + b_1y + \cdots),$$

where the first four factors, when separately equated to zero, give the respective equations of the boundaries.

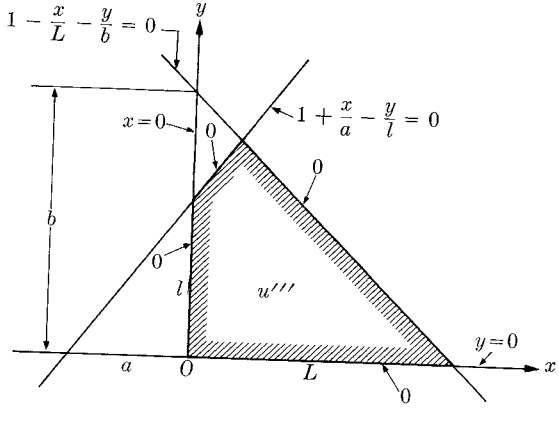


FIG. 8–18

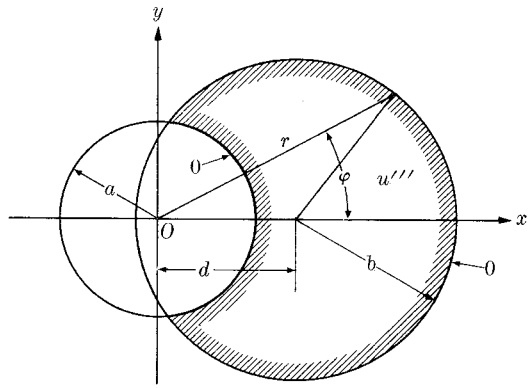


FIG. 8–19

Similarly, the Ritz profile of the steady two-dimensional problem of Fig. 8–19 may be constructed as

$$\theta(x, y) = (a^2 - x^2 - y^2)[b^2 - (x - d)^2 - y^2](a_0 + a_1 x + b_1 y^2 + \cdots),$$

where the first two factors are obtained from the equations of the circles. The same problem may also be expressed in terms of polar coordinates in the form

$$\theta(r, \varphi) = (a - r)(b^2 - d^2 - r^2 + 2rd \cos \varphi)(a_0 + a_1 r \cos \varphi + \cdots).$$

In general, the Ritz procedure is powerful for handling problems involving irregular geometry with boundaries kept at a constant temperature, say zero; these problems are not suitable to exact analytical methods. However, when the boundary conditions are other than those of constant temperature, the Ritz profile often becomes difficult to contruct. The Kantorovich method is also inconvenient for these problems.

(b) *The Ritz and Kantorovich profiles of problems involving regular geometry, and nonzero temperature on the boundaries.* By means of the principle of superposition, the Ritz profile of these problems may be expressed in terms of problems having zero temperature on the boundaries plus one or more suitable terms satisfying the nonzero temperatures on the boundaries.

Consider, for example, that the temperature of one boundary of a steady two-dimensional problem is specified, say by a parabola, while the remaining boundaries are kept at zero temperature (Fig. 8–20). Assuming $\theta(x, y)$ as the temperature distribution to be found, a new function $\psi(x, y)$ is defined such that the difference $\theta(x, y) - \psi(x, y)$ satisfies the boundary conditions of the problem. The simplest possible profile for this difference is

$$\theta(x, y) - \psi(x, y) = (x/L)[1 - (y/l)^2]T_0.$$

Here $\psi(x, y)$, satisfying the homogeneous boundary conditions

$$\psi(x, \pm l) = \psi(0, y) = \psi(L, y) = 0,$$

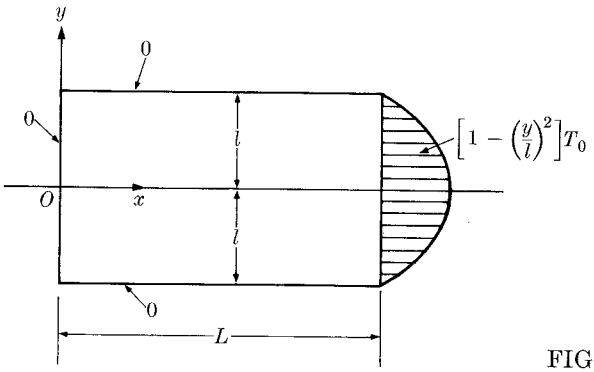


FIG. 8–20

can readily be determined. The Ritz profile of $\psi(x, y)$ is

$$\psi(x, y) = x(L - x)(l^2 - y^2)(a_0 + a_1x + b_1y^2 + \cdots).$$

Unlike the Ritz profile, the Kantorovich profile requires no superposition, and can be directly expressed in the form

$$\theta(x, y) = T_0[1 - (y/l)^2][X_0(x) + y^2X_1(x) + \cdots],$$

subject to boundary conditions

$$\begin{aligned} X_0(0) &= 0, & X_0(L) &= 1, \\ X_1(0) &= 0, & X_1(L) &= 0, \\ &\vdots & &\vdots \end{aligned}$$

On the other hand, if the nonhomogeneous temperature is constant (Fig. 8–21), the boundaries of the problem cannot be represented adequately by a finite number of approximations. Here, expressing the uniform temperature by an orthogonal (or nonorthogonal) set and leaving the x-direction to be determined, we may conveniently employ the Kantorovich method. For

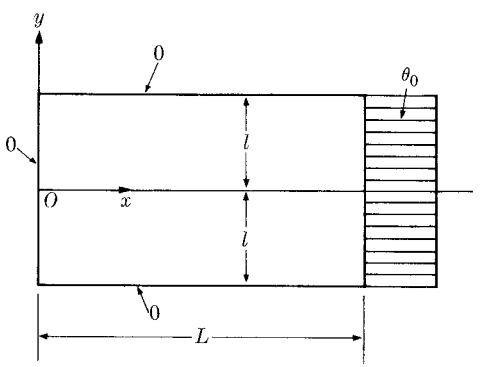


FIG. 8–21

example, using the orthogonal set which is the result of the characteristic-value problem

$$Y'' + \lambda^2 Y = 0, \qquad Y'(0) = 0, \qquad Y(l) = 0,$$

we may write the Kantorovich profile in the form

$$\frac{\theta(x, y)}{\theta_0} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} X_n(x) \cos \lambda_n y, \quad \lambda_n l = (2n + 1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots,$$

with the appropriate boundary conditions

$$X_n(0) = 0, \qquad X_n(L) = 1;$$

this leads to the *exact* solution of the problem.

The Ritz procedure, requiring the superposition of two profiles, say,

$$\frac{\theta(x, y)}{\theta_0} = \psi(x, y) + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda_n l)} \left(\frac{x}{L}\right)^{n+1} \cos \lambda_n y, \quad \lambda_n l = (2n + 1)\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots,$$

is tedious. Here $\psi(x, y)$ vanishes on the boundaries of the problem.

(c) *The Ritz and Kantorovich profiles of problems involving regular geometry and heat transfer to the ambient.* In terms of a product solution the problem is reduced to the selection of one-dimensional problems in each direction of the problem. For example, a Ritz profile for Example 2–1 with a moderate h (see Fig. 8–14) may readily be written in the form

$$\theta(x, y) = \left[\left(1 + \frac{2}{hL/k}\right) L^2 - x^2\right]\left[\left(1 + \frac{2}{hl/k}\right) l^2 - y^2\right] a_0 + (L - x)^2 (l - y)^2 \times [a_1 + a_2 x + a_3 x^2 + \cdots + b_2 y + b_3 y^2 + \cdots + c_3 xy + c_4 x^2 y + c_5 x y^2 + \cdots], \qquad x \geq 0, \quad y \geq 0,$$

and a Kantorovich profile may be written in the form

$$\theta(x, y) = \left[\left(1 + \frac{2}{hl/k}\right) l^2 - y^2\right] X_0(x) + (l - y)^2 [X_1(x) + y X_2(x) + y^2 X_3(x) + \cdots], \qquad y \geq 0,$$

together with the boundary conditions

$$X_0'(0) = 0, \qquad X_0'(L) + \frac{h}{k} X_0(L) = 0,$$

$$X_1'(0) = 0, \qquad X_1'(L) = X_1(L) = 0,$$

$$\vdots \qquad\qquad \vdots$$

Profiles for irregular geometry with specified temperature or heat flux, or heat transfer to the ambient, are often difficult to construct, and need further investigation.

The procedure for steady two-dimensional cartesian problems may readily be extended to steady three-dimensional problems. Similarly, steady one-, two-, and three-dimensional problems of cylindrical and spherical geometry require no additional basic knowledge.

8–11. Unsteady Problems

As with steady problems, by taking the governing differential equation of an unsteady problem as the Euler equation of the desired variational formulation we readily obtain

$$\int_{t_1}^{t_2} \int_{\mathcal{V}} \left(\nabla^2 \theta - \frac{1}{a} \frac{\partial \theta}{\partial t} \right) \delta\theta \, d\mathcal{V} \, dt = 0, \tag{8–110}$$

where $d\mathcal{V}$ is the differential volume element. Although the proof is too involved for us to consider here, it is possible to establish the equality of Eq. (8–110) to the variational formulation, $\delta I = 0$. Thus Eq. (8–110) will be used in this section as an equivalent form of the variational formulation of unsteady problems.

The difficulty in determining the timewise variation of the temperature makes the Ritz procedure unsuitable to unsteady problems. However, if we leave the time direction undetermined, the Kantorovich method may conveniently be used. We assume a profile,*

$$\theta(\mathcal{V}, t) = \theta[\mathcal{V}, \tau_0(t), \tau_1(t), \ldots, \tau_n(t)] \tag{3–211}$$

which is specified in space, but depends on the unknown functions $\tau_0(t)$, $\tau_1(t), \ldots, \tau_n(t)$ in time. Inserting Eq. (3–211) into Eq. (8–110) gives

$$\int_{t_1}^{t_2} \int_{\mathcal{V}} \left(\nabla^2 \theta - \frac{1}{a} \frac{\partial \theta}{\partial t} \right) \left(\frac{\partial \theta}{\partial \tau_0} \delta\tau_0 + \frac{\partial \theta}{\partial \tau_1} \delta\tau_1 + \cdots + \frac{\partial \theta}{\partial \tau_n} \delta\tau_n \right) d\mathcal{V} \, dt = 0. \tag{8–111}$$

Since Eq. (8–111) is true for an arbitrary time interval (t_1, t_2), the integrand relative to time integration must vanish everywhere in this interval.† Thus we have

$$\int_{\mathcal{V}} \left(\nabla^2 \theta - \frac{1}{a} \frac{\partial \theta}{\partial t} \right) \left(\frac{\partial \theta}{\partial \tau_0} \delta\tau_0 + \frac{\partial \theta}{\partial \tau_1} \delta\tau_1 + \cdots + \frac{\partial \theta}{\partial \tau_n} \delta\tau_n \right) d\mathcal{V} = 0. \tag{8–112}$$

* See Section 3–10.

† This reasoning is identical to that following Eq. (2–138).

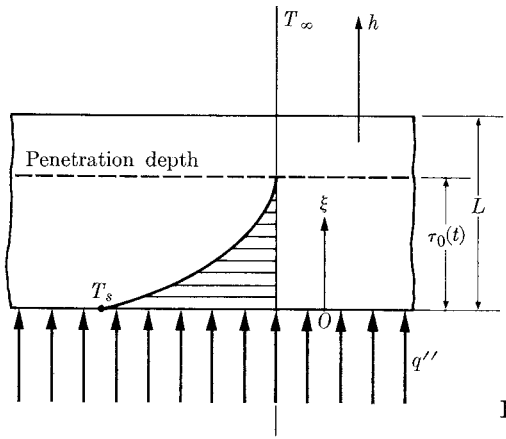


FIG. 8–22

Furthermore, since the variations $\delta\tau_0, \delta\tau_1, \ldots, \delta\tau_n$ are arbitrary, their coefficients must vanish in Eq. (8–112), giving the equations

$$\int_{\mathcal{V}} \left(\nabla^2\theta - \frac{1}{a}\frac{\partial\theta}{\partial t}\right)\frac{\partial\theta}{\partial\tau_i}\,d\mathcal{V} = 0, \qquad i = 0, 1, 2, \ldots, n. \tag{8–113}$$

After space integration, Eq. (8–113) yields $(n + 1)$ simultaneous ordinary differential equations in terms of the unknown functions $\tau_0(t), \tau_1(t), \ldots, \tau_n(t)$.

Example 8–8. Let us find a first-order approximate solution for the problem of Example 2–3.

Here Eq. (8–113) reduces to

$$\int_0^L \left(\frac{\partial^2\theta}{\partial x^2} - \frac{1}{a}\frac{\partial\theta}{\partial t}\right)\frac{\partial\theta}{\partial\tau_0}\,dx = 0 \tag{8–114}$$

with the assumption that the time dependence is adequately described by a single function τ_0. Again the time domain is conveniently divided into two regions.* In the first domain the penetration depth $\tau_0(t)$ is less than the thickness of the plate. The corresponding profile, in ξ measured from the bottom of the plate (Fig. 8–22), may be written in the form

$$\theta(\xi, t) = \frac{q''}{2k}\tau_0\left(1 - \frac{\xi}{\tau_0}\right)^2. \tag{8–115}$$

[Why can we not use the moving coordinate axis of Eq. (2–3) for the present

* See the integral formulation of Example 2–3.

problem?] Introducing Eq. (8–115) into Eq. (8–114) and rearranging gives

$$\int_0^{\tau_0} \left\{ \frac{1}{\tau_0} - \frac{1}{2a}\left[\left(1 - \frac{\xi}{\tau_0}\right)^2 + 2\left(1 - \frac{\xi}{\tau_0}\right)\frac{\xi}{\tau_0}\right]\frac{d\tau_0}{dt} \right\} \times \left[\left(1 - \frac{\xi}{\tau_0}\right)^2 + 2\left(1 - \frac{\xi}{\tau_0}\right)\frac{\xi}{\tau_0}\right] d\xi = 0.$$

The integration results in

$$d\tau_0^2 = 5a\, dt.$$

The solution of this equation which satisfies the initial condition $\tau_0(0) = 0$ is

$$\tau_0(t) = (5at)^{1/2}. \tag{8–116}$$

Replacing τ_0 by L in Eq. (8–116), we find that the penetration time of the heat flux to the upper surface is*

$$t_0 = L^2/5a. \tag{8–117}$$

The profile constructed from Fig. 2–42 for the second time domain of Example 2–3,

$$\theta(x, t) = \left(\frac{q''}{h} - \tau_1\right)\frac{hx^2}{2kL} + \left(1 + \frac{hx}{k}\right)\tau_1, \tag{2–193}$$

may also be used in the present problem. Thus, introducing Eq. (2–193) into Eq. (8–114), we obtain

$$\int_0^L \left[\frac{1}{kL}(q'' - h\tau_1) - \frac{1}{a}\left(1 + \frac{hx}{k} - \frac{hx^2}{2kL}\right)\frac{d\tau_1}{dt}\right]\left(1 + \frac{hx}{k} - \frac{hx^2}{2kL}\right) dx = 0.$$

The integration gives the differential equation

$$\left[1 + \frac{2}{3}\left(\frac{hL}{k}\right) + \frac{2}{15}\left(\frac{hL}{k}\right)^2\right]\frac{d\tau_1}{dt} + \frac{h}{\rho cL}\left(1 + \frac{hL}{3k}\right)\left(\tau_1 - \frac{q''}{h}\right) = 0. \tag{8–118}$$

The solution of Eq. (8–118), subject to $\tau_1(t_0) = 0$, is

$$\tau_1(t) = \frac{q''}{h}\left\{1 - \exp\left[-\frac{(1 + hL/3k)h(t - t_0)/\rho cL}{1 + (2/3)(hL/k) + (2/15)(hL/k)^2}\right]\right\}.$$

Finally, the complete solution of the problem expressed in terms of ξ is found

* Compare Eqs. (8–116) and (8–117) with Eqs. (2–184) and (2–186), respectively.

to be

$$\theta(\xi, t) = \left(\frac{q''}{2k}\right)(5at)^{1/2}\left[1 - \frac{\xi}{(5at)^{1/2}}\right]^2, \qquad \text{when} \qquad 0 \leq t \leq 0, \tag{8–119}$$

$$\frac{\theta(x, t)}{q''L/k} = \frac{1}{2}\left(1 - \frac{\xi}{L}\right)^2 + \left\{\frac{k}{hL} + \frac{1}{2}\left[1 - \left(\frac{\xi}{L}\right)^2\right]\right\}$$

$$\times \left\{1 - \exp\left[-\frac{(1 + hL/3k)h(t - t_0)/\rho cL}{1 + (2/3)(hL/k) + (2/15)(hL/k)^2}\right]\right\}, \qquad \text{when} \quad t \geq t_0. \tag{8–120}$$

Example 8–9. Let us find the solution of the unsteady problem associated with Example 8–3. Initially the system is at the ambient temperature. Then the upper disk suddenly assumes the angular velocity ω.

The dimensionless form of Eq. (8–114) suitable to our problem is

$$\int_0^1 \left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\theta}{\partial\rho}\right) - \mu^2\rho\theta + \nu\rho^2 - \frac{\partial\theta}{\partial t}\right]\frac{\partial\theta}{\partial\tau_i}\,d\rho = 0, \qquad i = 0, 1, 2, \ldots \tag{8–121}$$

where $t = at^*/R^2$, t^* being the actual time; the rest of the notation is the same as that of Example 8–3.

Assuming an instantaneous temperature distribution identical to the steady temperature given by Eq. (8–46),† and replacing the parameters a_0, a_1 with the time-dependent functions $a_0(t)$, $a_1(t)$, we may write the unsteady profile of the problem in the form

$$\theta(\rho, \tau) = a_0(t) + (3\rho^2 - 2\rho^3)a_1(t). \tag{8–122}$$

Introducing Eq. (8–122) into Eq. (8–121) yields

$$\int_0^1 \left[6(2\rho - 3\rho^2)a_1 - \mu^2\rho[a_0 + (3\rho^2 - 2\rho^3)a_1] + \nu\rho^2 - \rho\left[\frac{da_0}{dt} + (3\rho^2 - 2\rho^3)\frac{da_1}{dt}\right]\right]d\rho = 0,$$

$$\int_0^1 \left[6(2\rho - 3\rho^2)a_1 - \mu^2\rho[a_0 + (3\rho^2 - 2\rho^3)a_1] + \nu\rho^2 - \rho\left[\frac{da_0}{dt} + (3\rho^2 - 2\rho^3)\frac{da_1}{dt}\right]\right](3\rho^2 - 2\rho^3)\,d\rho = 0.$$

† See also Eqs. (8–55) and (8–56).

These integrals give the simultaneous differential equations

$$\frac{da_0}{dt} + \frac{7}{10}\frac{da_1}{dt} + \mu^2 a_0 + \frac{7}{10}\mu^2 a_1 = \frac{2}{3}\nu, \tag{8–123}$$

$$\frac{7}{20}\frac{da_0}{dt} + \frac{2}{7}\frac{da_1}{dt} + \frac{7}{20}\mu^2 a_0 + \left(\frac{2}{7}\mu^2 + \frac{3}{5}\right)a_0 = \frac{4}{15}\nu. \tag{8–124}$$

By means of the linear operator $D \equiv d/dt$, Eqs. (8–123) and (8–124) may be written in the operational forms

$$(D + \mu^2)a_0 + \tfrac{7}{10}(D + \mu^2)a_1 = \tfrac{2}{3}\nu,$$
$$\tfrac{7}{20}(D + \mu^2)a_0 + \tfrac{2}{7}(D + \mu^2 + \tfrac{21}{10})a_1 = \tfrac{4}{15}\nu.$$

Solving these equations for a_0 and a_1, replacing D by d/dt, and solving the resulting differential equations gives

$$a_0(t) = Ae^{-\mu^2 t} + Be^{-(\mu^2+840/57)t} + \frac{16\nu}{171\mu^2}\left(\frac{\mu^2 + 105}{\mu^2 + 840/57}\right), \tag{8–125}$$

$$a_1(t) = Ce^{-\mu^2 t} + De^{-(\mu^2+840/57)t} + \frac{140\nu}{171(\mu^2 + 840/57)}. \tag{8–126}$$

To determine the relationship which exists among the constants A, B, C, and D, Eqs. (8–125) and (8–126) are inserted, for example, into Eq. (8–124). Thus we have

$$A = 0, \qquad B = -\tfrac{10}{7}D.$$

Introducing A and B into Eqs. (8–125) and (8–126), and using the initial conditions $a_0(0) = 0$, $a_1(0) = 0$ results in

$$C = -\frac{84\nu(9/5 + 14/\mu^2)}{171(\mu^2 + 840/57)}, \qquad D = \left(\frac{7}{10}\right)\frac{16\nu(\mu^2 + 105)}{171\mu^2(\mu^2 + 840/57)}.$$

Thus inserting the values of A, B, C, and D into Eqs. (8–125) and (8–126), and the result into Eq. (8–122) yields the approximate Kantorovich profile

$$\frac{\theta(\rho, \tau)}{\nu} = \frac{16(\mu^2 + 105)}{171\mu^2(\mu^2 + 840/57)}[1 - e^{-(\mu^2+840/57)t}] + \frac{140(3\rho^2 - 2\rho^3)}{171(\mu^2 + 840/57)}$$
$$\times\left[1 - \frac{3}{5}\left(\frac{9}{5} + \frac{14}{\mu^2}\right)e^{-\mu^2 t} + \frac{2}{5}\left(\frac{1}{5} + \frac{21}{\mu^2}\right)e^{-(\mu^2+840/57)t}\right]. \tag{8–127}$$

For large values of time the foregoing Kantorovich profile approaches the steady Ritz profile given by Eq. (8–47), as expected.

8–12. Some Definite Integrals

Here, for convenience, we list the results of some definite integrals appearing frequently in problems of variational calculus:

$$\int_0^l x^m(l-x)^n\,dx = \frac{m!n!}{(m+n+1)!}\,l^{(m+n+1)}, \tag{8–128}$$

$$\int_0^l (l^2-x^2)^m\,dx = \frac{(2m)!!}{(2m+1)!!}\,l^{(2m+1)}, \tag{8–129}$$

$$\int_0^l x^{2m+1}(l^2-x^2)^n\,dx = \frac{(2m)!!(2n)!!}{(2m+2n+2)!!}\,l^{(2m+2n+2)}, \tag{8–130}$$

$$\int_0^l x^{2m+2}(l^2-x^2)^n\,dx = \frac{(2m+1)!!(2n)!!}{(2m+2n+3)!!}\,l^{(2m+2n+3)}, \tag{8–131}$$

$$\int_0^{\pi/2} \sin^m\phi\,d\phi = \int_0^{\pi/2} \cos^m\phi\,d\phi = \begin{cases} \dfrac{(m-1)!!}{m!!}, & m \text{ odd}, \\ \dfrac{(m-1)!!}{m!!}\,\dfrac{\pi}{2}, & m \text{ even}, \end{cases} \tag{8–132}$$

$$\int_0^{\pi/2} \sin^m\phi\cos^n\phi\,d\phi = \begin{cases} \dfrac{(m+n-1)!!}{(m+n)!!}, & m \text{ or } n \text{ odd}, \\ \dfrac{(m+n-1)!!}{(m+n)!!}\,\dfrac{\pi}{2}, & m \text{ and } n \text{ even}, \end{cases} \tag{8–133}$$

where, by definition,

$$0! = 1, \qquad m! = m(m-1)(m-2)\cdots,$$
$$0!! = 1, \qquad m!! = m(m-2)(m-4)\cdots$$

For integrals other than those listed above, the reader is referred to integral tables.

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